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Spontaneous Color Polarization as A *Modus Originis* of the Dynamic Aether

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Abstract: We suggest the phenomenological model of emergence of the dynamic aether as a result of decay of the SU(N) symmetric field configuration containing the multiplet of vector fields. The scenario of the transition to the dynamic aether, which is characterized by one unit timelike vector field that is associated with the aether velocity, is based on the idea of spontaneous color polarization analogous to the spontaneous electric polarization in ferroelectric materials. The mechanism of spontaneous color polarization is described in the framework of anisotropic cosmological model of the Bianchi-I type; it involves consideration of the idea of critical behavior of the eigenvalues of the tensor of color polarization in the course of the Universe accelerated expansion. The interim stage of transition from the color aether to the canonic dynamic aether takes the finite period of time, the duration of which is predetermined by the phenomenologically introduced critical value of the expansion scalar.

Keywords: dynamic aether; multiplet of vector fields

1. Introduction

The term dynamic aether was introduced in the framework of the Einstein-aether theory, which was formulated in the beginning of 21 century [1–3]. This theory operates with a timelike unit dynamic vector field U^i, which is interpreted as the velocity four-vector of some cosmic substratum indicated as the aether. The color dynamic aether, which is considered in the presented work, appears in a SU(N) symmetric generalization of the dynamic aether model. The motives of such an extension include physical and mathematical aspects; we briefly consider them in the context of our study.

1.1. Basic Elements of the Einstein-Aether Theory and the Problem of Identification of the Coupling Constants

The Einstein–aether theory belongs to the vector-tensor branch of the science, known as Modified Gravity (see, e.g., [4,5] for details of its classification); it is based on the action functional

$$S_{(\mathrm{EA})} = \int \frac{d^4x\sqrt{-g}}{2\kappa}\left[R + 2\Lambda + \lambda(g_{ik}U^iU^k - 1) + \mathcal{K}\right] , \qquad (1)$$

where R is the Ricci scalar; Λ is the cosmological constant. The function λ is the Lagrange multiplier and the term $\lambda(g_{ik}U^iU^k - 1)$ has to provide the normalization condition $U^kU_k = 1$. One of the consequences of this normalization condition is the differential constraint $U_k\nabla_jU^k = 0$. The scalar $\mathcal{K} = K^{ab}_{mn}\nabla_aU^m\nabla_bU^n$ is the kinetic term quadratic in the covariant derivative ∇_aU^m. According to the idea of the authors of the works [1–3], the constitutive tensor K^{ab}_{mn} has the form

$$K^{ab}_{mn} = C_1g^{ab}g_{mn} + C_2\delta^a_m\delta^b_n + C_3\delta^a_n\delta^b_m + C_4U^aU^bg_{mn} , \qquad (2)$$

i.e., it contains not only the basic geometric objects: the metric tensor g_{mn} and the Kronecker tensor δ^i_k, but the four-vector U^i also. The decomposition (2) is equipped by four coupling constants C_1, C_2, C_3, C_4 introduced phenomenologically. When the term $\mathcal{K}$ is considered to be free of the four-vector U^i, one has to put $C_4 = 0$. When the four-vector U^i is the necessary element of the constitutive tensor K^{ab}_{mn}, one can consider, formally speaking, the terms $C_5 U_m U_n g^{ab}$, $C_6 U^a U_n \delta^b_m$, $C_7 U^b U_n \delta^a_m$, $C_8 U^a U_m \delta^b_n$, $C_9 U^b U_m \delta^a_n$, $C_{10} U^a U^b U_m U_n$, as the additional elements of the Lagrangian; however, due to the consequence $U_k \nabla_j U^k = 0$ all of these terms disappear from the master equations, and the coupling constants $C_5, ..., C_{10}$ become hidden. In other words, the canonic variant of the Einstein–aether theory contains four parameters. The constraints on the quartet of coupling constants C_1, C_2, C_3, and C_4 were formulated based on Parameterized Post-Newtonian (PPN) formalism [6]; the details can be found, e.g., in the status report [3]).

In the context of the coupling constant estimation, we would like to mention an interesting example, which is indicated as the Maxwell-like sub-model. It appears when $C_1 = -C_3$, $C_2 = 0$, $C_4 = 0$, so that the scalar $\mathcal{K}$ contains the square of the skew-symmetric term $\mathcal{F}_{ik} = \nabla_i U_k - \nabla_k U_i$ only, i.e., $\mathcal{K} = \frac{1}{2} C_1 \mathcal{F}_{ik} \mathcal{F}^{ik}$. Of course, the covariant derivative $\nabla_i U_k$ has both symmetric and skew-symmetric parts, but the only skew-symmetric element appears in the scalar $\mathcal{K}$ due to the specific choice of the coupling parameters. In 2017, the events, encoded as GW170817 and GRB 170817A [7], changed the status of the constraint $C_1 = -C_3$ from a hypothesis to the experimental fact. Based on the observation of the binary neutron star merger, it was established that the ratio of the velocities of the gravitational and electromagnetic waves differs from one by the quantity about 10^{-15} (to be more precise, $1 - 3 \times 10^{-15} < \frac{v_{\rm gw}}{c} < 1 + 7 \times 10^{-16}$). Because the square of the velocity of the tensorial spin-2 aether mode was calculated in [2] to be $S^2_{(2)} = \frac{1}{1-(C_1+C_3)}$, one can estimate the sum of the parameters $C_1 + C_3$, as follows: $-6 \times 10^{-15} < C_1 + C_3 < 1.4 \times 10^{-15}$. This result admits three variants of interpretation. First, when $C_1 + C_3 \equiv 0$ and thus $S_{(2)} \equiv 1$, we can additionally require only that the squares of the velocities of the vectorial spin-1 and of the scalar spin-0 aether modes are non-negative, thus providing the inequalities $S^2_{(1)} = \frac{C_1}{C_1+C_4} \geq 0$ and $S^2_{(0)} = \left[\frac{2}{C_1+C_4} - 1\right] / \left[3 + \frac{2}{C_2}\right] \geq 0$, respectively. Second, if we assume that $1 - 3 \times 10^{-15} < \frac{v_{\rm gw}}{c} < 1$, i.e., the spin-2 aether wave mode is subluminal, we have to take into account the possible gravitational Cherenkov effect [8,9]). As it was shown in [10,11], the corresponding anomalous acceleration of cosmic particles is absent, so the spin-1 and spin-0 aether modes should be superluminal, and we have to consider additional inequalities $S_{(1)} > 1$ and $S_{(0)} > 1$, providing the constraint $0 < C_2 < 0.095$ [10]. Third, if we assume that $1 < \frac{v_{\rm gw}}{c} < 1 + 7 \times 10^{-16}$, i.e., the spin-2 mode is superluminal, the argument concerning the gravitational Cherenkov effect can not be used, and the requirements that the phase velocities of the spin-1 and spin-0 modes are bigger than 1, are not necessary. In this case, the constraint for C_2 is $-\frac{2}{27} < C_2 < \frac{2}{21}$ (see [10,12]).

We see that the recent success in the gravitational wave experiments has provided the progress in the constraining of the Einstein–aether theory. We can hope that the predictions based on the solutions describing black holes [13–15], static spherically symmetric objects, neutron stars and binary systems [16–18], universal horizon,s and thermodynamics of the Einstein–aether black holes (see, e.g., [19–21]), will also be tested in the nearest future and will introduce new constraints on the coupling constants.

1.2. Einstein–Aether Theory: Historical Motives and Related Models

The Einstein-aether theory [1–3] realizes two remarkable ideas. The first idea is connected with the assumption that there exists a preferred frame of reference advocated by Nordtvedt and Will [6,22,23]. Indeed, if one has a timelike global unit continuous vector field U^i, one can reconstruct the preferred coordinate system, for which $\frac{dx^i}{ds} = U^i$; in this context, the vector field U^i can be interpreted as the velocity four-vector associated with the flow of some medium (e.g., the aether). The second remarkable idea, which was suggested by Dirac in [24–26], is that the unit timelike vector field can play the dual role: of the vector potential, which satisfies a special gauge condition, and of the velocity four-vector of a charged particle flow. This Dirac's idea can be extended as follows: let a timelike vector field potential $\mathcal{U}^i$ as a source of the gravity field exist, but let its norm be not equal to one. In this case, one can introduce the scalar field as the modulus of this vector field, $\mathcal{U} = \sqrt{g_{mn}\mathcal{U}^m\mathcal{U}^n}$, and the normalized unit four-vector $V^i \equiv \frac{\mathcal{U}^i}{\mathcal{U}}$. Thus, we deal with a version of the scalar-vector-tensor theory of gravity, based on the interacting trio $\{\mathcal{U}, V^j, g_{ik}\}$.

There exist a class of models, which can be indicated as the related to the Einstein-aether theory. As an example of such model, we would like to consider the so-called bumblebee model (see, e.g., [27–30]). Such a model differs from the Einstein–aether theory by one term only: it is the term $\lambda(g_{ik}B^iB^k \pm b^2)^2$ introduced into the Lagrangian instead of the term $\lambda(g_{ik}U^iU^k - 1)$. Now, λ is not a Lagrange multiplier, it is a phenomenological constant; and the new term gives the fourth order potential of the Higgs type. The vector field B^k in this theory is not normalized, and the constant $|b|$ is not a norm, but it is the vacuum expectation value corresponding to the minimum of the potential.

1.3. Generalizations and Extensions of the Einstein-Aether Theory

With any approach, the interpretation of the vector field U^i as the aether velocity four-vector inevitably revives the discussions concerning the taboo that is associated with the Lorentz invariance violation (see, e.g., [31–33]). Breaking this taboo, many authors consider the modifications of the Einstein–aether theory; we distinguish between generalizations and extensions of this theory. The generalization of the first type is based on the introduction of the nonlinear term $F(\mathcal{K})$ instead of the linear term $\mathcal{K}$ into the Lagrangian, $\mathcal{K} \to F(\mathcal{K})$ (see, e.g., [34,35]). Clearly, the idea of such nonlinear generalization is inspired by the $f(R)$-extension of the theory of gravity, in the framework of which the Ricci scalar R is replaced by the nonlinear function $R \to f(R)$ (see, e.g., [4,5]). The generalization of the second type is predetermined by modifications in the gravitational part of the Lagrangian; there are modifications of the Horava–Lifshitz type (see, e.g., [36,37]), the modifications based on the metric-affine gravity [38], on the $f(R)$, $f(G)$, $f(T)$, etc., models of gravity (see, e.g., [39]). The generalizations of the third type involve consideration of extra-dimensions [40] and the ideas of supersymmetry [41–43].

We speak about extensions of the Einstein–aether theory, when one already has the unit vector field U^i, and one adds new terms into the action functional, which describe the scalar, pseudoscalar, electromagnetic, gauge, etc., fields. The scalar extension of the Einstein–aether theory was motivated by cosmological applications, (see, e.g., [44–49]), in particular, to precise some details of inflation, dynamics of perturbations, and the large-scale structure formation. In order to describe the impact of the scalar field φ on the unit vector field U^k, the authors of [44] suggested to modify the constitutive tensor K^{ab}_{mn} (2) by introduction of the functions $\beta_1(\varphi)$, $\beta_2(\varphi)$, $\beta_3(\varphi)$, $\beta_4(\varphi)$ instead of constants C_1, C_2, C_3, C_4, respectively. The Einstein–Maxwell–aether theory as an electromagnetic extension of the Einstein–aether theory, was established in [50] and applied to the problem of birefringence in [51]. The Einstein–Yang–Mills–aether theory as a SU(N) generalization of the U(1) symmetric Einstein–Maxwell–aether theory is considered in the work [52]. Pseudoscalar (axionic) extension of the Einstein–aether theory was presented in the work [53]. The next natural step in the development of the Einstein–aether theory was the combination of the electromagnetic and pseudoscalar modifications indicated as axionic extension of the Einstein–Maxwell–aether theory [54–56].

1.4. What Is the Aim of This Work?

In the paper [52] we proposed an idea of the color aether, which is based on the introduction of the SU(N) symmetric multiplet of vector fields $U^i_{(a)}$. The subscript (a) denotes the group index in the adjoint representation of the SU(N) group and this Latin index runs over the values $\{(1),(2),..(N^2{-}1)\}$. If all of the vectors $U^i_{(a)}$ from this multiplet become parallel, i.e., when $U^i_{(a)}{=}q_{(a)}U^i$, and, if U^i is the unit timelike vector field, we happen to be faced with the version of the dynamic aether, supplemented by the SU(N) symmetric multiplet of scalar fields $\{q_{(a)}\}$. Clearly, $q_{(a)}$ are the scalars from the point of view of spacetime transformations, but the corresponding object with upper index, $q^{(a)} = G^{(a)(b)}q_{(b)}$, can be considered as vectors in the N^2-1 dimensional group space with the metric $G^{(a)(b)}$ (color vectors, for short). When we deal with the SU(N) symmetric multiplet of vector fields, we have to use the self-consistent formalism, which includes the Yang–Mills gauge fields with the potentials $A^{(a)}_k$, the gauge covariant derivative $\hat{D}^{(a)}_m$, etc. This full-format SU(N) symmetric formalism is elaborated and presented in the work [52] as the extension of the U(1) symmetric formalism that is discussed in [50] in the framework of Einstein–Maxwell–aether theory.

In order to establish the phenomenological model of the transition from the color aether to the dynamic aether, we exploit the known analogy with the spontaneous polarization in the ferroelectric materials [57,58]; the term spontaneous means that the electric polarization is induced by variations of temperature or pressure, not by the external electric field. We keep in mind three principal details that appeared in both models. The first detail is connected with the character of evolution. In the theory of ferroelectricity, one can associate the evolution with the temperature decrease: high temperature relates to the symmetric phase without polarization, low temperature corresponds to the polarized dissymmetric phase. When we consider the evolution of the color system in the expanding Universe, the effective temperature is also dropping, providing the transition from the symmetric phase to the dissymmetric one. The second detail is connected with the Curie temperature T_C: below this critical value, $T < T_C$, the spontaneous electric polarization appears, and, thus, T_C indicates the temperature of the ferroelectric phase transition. Similarly, we can introduce the critical value of the expansion scalar Θ_*, which relates to the cosmological phase transition taking place at the cosmological time t_*. The third formal accordance is based on the analogy between the electric domain structure in ferroelectric (chaotic in the symmetric phase and aligned in the dissymmetric one) and the anisotropy of the group space that is associated with the color vector fields (multi-axial structure in the symmetric phase and uni-axial in the dissymmetric one). In other words, we assume that, in the early Universe, the phase transition took place, which transformed the SU(N) symmetric multiplet of vector fields to the bundle of vectors, which happen to be parallel in the group space. Below, we describe the mathematical details of the phenomenological model of the spontaneous color polarization.

This paper is organized, as follows. In Section 2, we recall the basic elements of the SU(N) symmetric field formalism and the corresponding master equations. In Section 3 we discuss the possible mechanism of transformation of the color aether, equipped by the SU(N) symmetric multiplet of the vector fields $\left\{U^i_{(a)}\right\}$, to the canonic dynamic aether, which possesses one unit vector field U^i. Section 4 contains the discussion and conclusion.

2. The Formalism

2.1. Basic Elements of the Theory

We follow the book [59] and use the Hermitian traceless generators of the SU(N) group, $\mathbf{t}_{(a)}$, which satisfy the commutation relations

$$\left[\mathbf{t}_{(a)}, \mathbf{t}_{(b)}\right] = i f^{(c)}_{(a)(b)} \mathbf{t}_{(c)} \,, \tag{3}$$

where $f^{(c)}_{(a)(b)}$ are the structure constants of the SU(N) gauge group. The scalar product of the generators $\mathbf{t}_{(a)}$ and $\mathbf{t}_{(b)}$ introduces the metric in the group space, $G_{(a)(b)}$:

$$G_{(a)(b)} = \left(\mathbf{t}_{(a)}, \mathbf{t}_{(b)}\right) \equiv 2\mathrm{Tr}\ \mathbf{t}_{(a)}\mathbf{t}_{(b)} \,. \tag{4}$$

The quantities

$$f_{(c)(a)(b)} \equiv G_{(c)(d)} f^{(d)}_{(a)(b)} = -2i\mathrm{Tr}\left[\mathbf{t}_{(a)}, \mathbf{t}_{(b)}\right] \mathbf{t}_{(c)} \tag{5}$$

are antisymmetric with respect to transposition of any two indices. The Yang-Mills field potential $\mathbf{A}_m$ and the Yang-Mills field strength $\mathbf{F}_{ik}$ are defined as follows:

$$\mathbf{A}_m = -i\mathcal{G}\mathbf{t}_{(a)}A^{(a)}_m \,, \quad \mathbf{F}_{mn} = -i\mathcal{G}\mathbf{t}_{(a)}F^{(a)}_{mn} \,, \tag{6}$$

where $\mathcal{G}$ is the coupling constant, and the multiplets of real fields $A^{(a)}_i$ and $F^{(a)}_{ik}$ are connected as follows:

$$F^{(a)}_{mn} = \nabla_m A^{(a)}_n - \nabla_n A^{(a)}_m + \mathcal{G}f^{(a)}_{\ (b)(c)} A^{(b)}_m A^{(c)}_n \,. \tag{7}$$

Here and below ∇_m is a covariant derivative. The tensors $F^{ik}_{(a)}$ has its dual elements

$$^*F^{ik}_{(a)} = \frac{1}{2}\epsilon^{ikls}F_{ls(a)} \,, \tag{8}$$

with universal Levi-Civita tensor ϵ^{ikls}. The multiplet of vector fields $U^{(a)}_m$ appears as the decomposition of the Hermitian quantity

$$\mathbf{U}_m = \mathbf{t}_{(a)}U^{(a)}_m \,. \tag{9}$$

The extended (gauge covariant) derivative $\hat{D}_m$ is defined as follows:

$$\hat{D}_m Q^{(a)\cdots}_{\cdots(d)} \equiv \nabla_m Q^{(a)\cdots}_{\cdots(d)} + \mathcal{G}f^{(a)}_{\cdot(b)(c)} A^{(b)}_m Q^{(c)\cdots}_{\cdots(d)} - \mathcal{G}f^{(c)}_{\cdot(b)(d)} A^{(b)}_m Q^{(a)\cdots}_{\cdots(c)} + \ldots \,, \tag{10}$$

where $Q^{(a)\cdots}_{\cdots(d)}$ is arbitrary tensor in the group space (see, e.g., [60]). For the vector fields this formula gives

$$\hat{D}_m U^{(a)}_n \equiv \nabla_m U^{(a)}_n + \mathcal{G}f^{(a)}_{\ (b)(c)} A^{(b)}_m U^{(c)}_n \,. \tag{11}$$

The dual tensor $^*F^{ik}_{(a)}$, due to the definitions (7) and (8), satisfies the relations

$$\hat{D}_k {}^*F^{ik}_{(a)} = 0 \,. \tag{12}$$

Finally, we introduce the multiplet of scalar fields $\Omega^{(a)}$, which forms the color vector

$$\Omega^{(a)} \equiv \hat{D}_m U^{(a)m} = \nabla_m U^{(a)m} + \mathcal{G}f^{(a)}_{\ (b)(c)} A^{(b)}_m U^{(c)m} \,, \tag{13}$$

and the scalar Ω based on the relationship

$$\Omega^2 = \Omega^{(a)}\Omega_{(a)} \,. \tag{14}$$

In the Abelian model of the aether with U(1) symmetry the scalar Ω coincides with the expansion scalar $\Theta \equiv \nabla_m U^m$.

2.2. The Ansatz

Our additional ansatz is that the vector fields satisfy the condition

$$G_{(a)(b)} U^{(a)}_m U^{(b)}_n g^{mn} = 1 \,. \tag{15}$$

This is the direct generalization of the normalization condition $g^{mn}U_m U_n = 1$, appeared in the canonic Einstein-aether theory, however, we have to keep in mind two new essential details. First, the set of vector fields $U^i_{(a)}$ can include s_0 timelike vectors with positive norms, s_1 spacelike vectors with negative norms, and $N^2-1-s_0-s_1$ null vectors. Clearly, the number s_0 can not be equal to zero, $s_0 \geq 1$, but this is the only general condition for the numbers s_0 and s_1. The second detail is the following: in contrast to the condition $U_k \nabla_j U^k = 0$, we see that generally $U^{(a)}_k \nabla_j U^k_{(b)} \neq 0$. This means, in particular, that the terms of the type $C_5 U^{(c)}_m U^{(d)}_n g^{ab}$ in the generalized constitutive tensor will not be hidden.

2.3. Action Functional and Master Equations

We use the action functional, which describes interaction of gauge, vector, and gravitational fields

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa} \left[R + 2\Lambda + \lambda \left(U^m_{(a)} U^{(a)}_m - 1 \right) + \mathcal{K}^{ijmn}_{(a)(b)} \hat{D}_i U^{(a)}_m \hat{D}_j U^{(b)}_n \right] + \frac{1}{4} F^{(a)}_{mn} F^{mn}_{(a)} \right\} . \quad (16)$$

The full-format version of the constitutive tensor $\mathcal{K}^{ijmn}_{(a)(b)}$ is presented in the paper [52]; that version contains 75 coupling constants, since now $U^{(a)}_k \nabla_j U^k_{(b)} \neq 0$, and the phenomenological approach requires to list all of the possible terms. In this work, based on the direct analogy with the canonic Einstein–aether theory and following the idea to study some illustrative toy model, we postulate the constitutive tensor to contain only four coupling constants:

$$\mathcal{K}^{ijmn}_{(a)(b)} = C_4 U^i_{(a)} U^j_{(b)} g^{mn} + G_{(a)(b)} \left[C_1 g^{ij} g^{mn} + C_2 g^{im} g^{jn} + C_3 g^{in} g^{jm} \right] . \quad (17)$$

Clearly, for the U(1) - symmetric theory the group indices disappear, and this constitutive tensor coincides with the Jacobson's one (2). On the other hand, this example explicitly shows that the constitutive tensor $\mathcal{K}^{ijmn}_{(a)(b)}$ is not generally proportional to the metric $G_{(a)(b)}$, and in this sense it does not possess the isotropy with respect to the group indices.

2.3.1. Master Equations for the Gauge Fields

Master equations appear as a result of variation of the action functional (16) with respect to four quantities: the Lagrange multiplier λ, the vector fields $U^j_{(a)}$, the gauge potential four-vector $A^{(b)}_i$, and space-time metric g^{pq}. The variation with respect to λ gives the normalization condition (15). Variation with respect to $A^{(a)}_i$ gives the extended Yang–Mills equations

$$\hat{D}_k F^{ik}_{(a)} = \Gamma^i_{(a)} , \quad (18)$$

where the color current $\Gamma^i_{(a)}$ is given by the formula

$$\Gamma^i_{(a)} = \frac{\mathcal{G}}{\kappa} f^{(d)}_{\ (c)(a)} U^{(c)}_k \mathcal{K}^{imkn}_{(d)(b)} \hat{D}_m U^{(b)}_n . \quad (19)$$

This quantity appears, when we fulfill the variation of the second term in the right-hand side of (11) with respect to $A^{(d)}_i$. From the physical point of view, the color current $\Gamma^i_{(a)}$ is induced by the interaction between the color vector field and non-Abelian gauge field; clearly, it disappears in the U(1) case, since the structure constants are equal to zero in that model.

2.3.2. Master Equations for the Vector Fields $U^k_{(a)}$

The variation with respect to $U^j_{(a)}$ yields the standard balance equation

$$\hat{D}_i \mathcal{J}^{ij}_{(a)} = \lambda \, U^j_{(a)} + \mathcal{I}^j_{(a)} , \quad (20)$$

where the color objects $\mathcal{J}^{ij}_{(a)}$ and $\mathcal{I}^j_{(a)}$ are introduced in analogy with the canonic aether theory [1]:

$$\mathcal{J}^{ij}_{(a)} = \mathcal{K}^{imjn}_{(a)(b)} \hat{D}_m U^{(b)}_n = C_1 \hat{D}^i U^j_{(a)} + C_2 g^{ij} \hat{D}_m U^m_{(a)} + C_3 \hat{D}^j U^i_{(a)} + C_4 U^i_{(a)} U^m_{(b)} \hat{D}_m U^{j(b)} , \quad (21)$$

$$\mathcal{I}^{j}_{(a)} = C_4 \hat{D}^j U_{m(a)} U^n_{(b)} \hat{D}_n U^{m(b)} \,. \tag{22}$$

Finally, the Lagrange multiplier can be found as

$$\lambda = U^{(a)}_j \left[\hat{D}_i \mathcal{J}^{ij}_{(a)} - \mathcal{I}^j_{(a)} \right] , \tag{23}$$

i.e., the set (20) contains $4N^2{-}5$ independent equations for the vector fields $U^k_{(a)}$ (instead of three ones in the U(1) model).

2.3.3. Master Equations for the Gravitational Field

The gravitational field is described by the set of equations

$$R_{pq} - \frac{1}{2} R g_{pq} = \Lambda g_{pq} + \lambda U^{(a)}_p U_{(a)q} + T_{pq} + T^{(\mathrm{YM})}_{pq} \,. \tag{24}$$

The stress-energy tensor of the color vector fields T_{pq} is of the following form:

$$T_{pq} = \frac{1}{2} g_{pq} \mathcal{K}^{ijmn}_{(a)(b)} \hat{D}_i U^{(a)}_m \hat{D}_j U^{(b)}_n + C_1 \left[\hat{D}_m U^{(a)}_p \hat{D}^m U_{q(a)} - \hat{D}_p U^{m(a)} \hat{D}_q U_{m(a)} \right] +$$
$$+ C_4 U^m_{(a)} \hat{D}_m U^{(a)}_p U^n_{(b)} \hat{D}_n U^{(b)}_q + G_{(a)(b)} \hat{D}^m \left[U^{(b)}_{(p} \mathcal{J}^{(a)}_{q)m} - \mathcal{J}^{(a)}_{m(p} U^{(b)}_{q)} - \mathcal{J}^{(a)}_{(pq)} U^{(b)}_m \right] . \tag{25}$$

Here, the parentheses (pq) denote the symmetrization with respect to the coordinate indices p and q. The symbol $T^{(\mathrm{YM})}_{pq}$ relates to the stress-energy tensor of the Yang–Mills field

$$T^{(\mathrm{YM})}_{pq} = \frac{1}{4} g_{pq} F^{(a)}_{mn} F^{mn}_{(a)} - F^{(a)}_{pm} F_{(a)qn} g^{mn} \,. \tag{26}$$

Thus, the total set of master equations is composed by (12), (18), (20) and (24).

2.4. The Tensor of Color Polarization

Based on the multiplet of the color fields $U^{(a)}_k$, we introduce an auxiliary quantity

$$H^{(a)(b)} \equiv g^{pq} U^{(a)}_p U^{(b)}_q \,. \tag{27}$$

It presents a set of scalar functions from the point of view of spacetime transformations, and is a symmetric real tensor in the group space. The quantity $H^{(a)(b)}$ can be indicated as the tensor of color polarization, the N^2-1 dimensional analog of the two-dimensional polarization tensor in optics (see, e.g., [61]). The components of this tensor depend on time and spatial coordinates. If we work with the matrix $H^{(a)}_{(b)} \equiv G_{(c)(b)} H^{(a)(c)}$ as with the algebraic quantity calculated in the fixed point of the spacetime, we can find $N^2{-}1$ eigenvalues, $\sigma_{\{\alpha\}}$, and the corresponding $N^2{-}1$ eigenvectors $q^{(a)}_{\{\alpha\}}$ (the subscript $\{\alpha\}$ indicates the serial number of the eigenvalue):

$$H^{(a)}_{(b)} q^{(b)}_{\{\alpha\}} = \sigma_{\{\alpha\}} q^{(a)}_{\{\alpha\}} \,. \tag{28}$$

Because the trace $H^{(a)}_{(a)}$ is equal to one due to the relationship

$$H^{(a)}_{(a)} = G_{(a)(b)} g^{mn} U^{(a)}_m U^{(b)}_n = 1 \,, \tag{29}$$

the sum of eigenvalues is equal to one,

$$\sum_{\alpha=1}^{N^2-1} \sigma_{\{\alpha\}} = 1\,. \tag{30}$$

One can decompose the tensor $H^{(a)(b)}$ into the series of products of eigenvectors:

$$H^{(a)(b)} = \sum_{\alpha=1}^{N^2-1} \sigma_{\{\alpha\}} q^{(a)}_{\{\alpha\}} q^{(b)}_{\{\alpha\}}\,. \tag{31}$$

There are two physically motivated limiting cases that are associated with the tensor of color polarization.

1. When all of the eigenvalues are equal to one another, and thus $\sigma_{\{\alpha\}} = \sigma = 1/(N^2-1)$, we deal with the color analog of the so-called natural light [61].
2. When all of the eigenvalues, except one, are equal to zero, we deal with the color analog of the linearly polarized light; in this case only one eigenvalue is non-vanishing, say, $\sigma_{\{1\}}$, and it is equal to one. We obtain now $H^{(a)(b)} = q^{(a)}_{\{1\}} q^{(b)}_{\{1\}}$, where $q^{(a)}_{\{1\}}$ is the corresponding eigenvector; the determinant of the matrix of this color tensor is equal to zero, the matrix is degenerated and, thus, the equality $\det H^{(a)(b)}{=}0$ can be considered as a criterion for recognition of the *color polarization*.

The crucial point of the presented model is the spontaneous process, which transforms the multiplet of color vector fields $U^i_{(a)}$ into the bundle of vectors $q_{(a)} U^i$, which are parallel in the group space. We can monitor this process while using the behavior of the eigenvalues $\sigma_{\{\alpha\}}$ of the matrix of color polarization $H^{(a)(b)}$. Our ansatz is that N^2-2 eigenvalues have the stepwise form

$$\sigma_{\{\alpha\}} = \eta\left[\Omega^*_{\{\alpha\}} - \Omega\right]\, \sigma^0_{\{\alpha\}} \left[\Omega^*_{\{\alpha\}} - \Omega\right]^{\mu}\,. \tag{32}$$

Here, $\eta[z]$ is the Heaviside function that is equal to zero, when the argument z is negative. The quantities $\Omega^*_{\{\alpha\}}$, where $\{\alpha\} = 2, 3...N^2-1$, are the critical values of the scalar Ω introduced in (13) and (14). These critical values can coincide or be different. The quantity $\sigma^0_{\{\alpha\}}$ is a constant, and the parameter μ plays the role of critical index; we assume that $\mu > 1$ providing the function $\sigma_{\{\alpha\}}$ and its derivative to be equal to zero at the critical moment. This stepwise representation means that there are critical moments of time $t^*_{\{\alpha\}}$, or equivalently, critical values of the scalar Ω, for which $N^2{-}2$ eigenvalues vanish and only one of them happens to be equal to one, since the total sum has to be unit. It may be a cascade of the eigenvalue disappearances (with many different critical times), or synchronized process (with only one critical time t_*), however the final of this process can be characterized by the degenerated matrix $H^{(a)(b)}$ with the rank equal to one. Thus, when $\Omega > \max\left\{\Omega^*_{\{\alpha\}}\right\}$, the color vector $q^{(a)}$ appears, which forms the polarization tensor $H^{(a)(b)} = q^{(a)} q^{(b)}$ according to the decomposition (31). The evolution of the color vector $q^{(a)}$ is described in the next Section.

3. Phenomenology of the Spontaneous Color Polarization

3.1. Basic Ansatz and Scheme of Analysis

3.1.1. On the Universe Evolution Scenario

We consider the toy model of the Universe expansion with three epochs, which we indicate as symmetric, interim transition, and dissymmetric, respectively, while keeping in mind the analogy with the theory of ferroelectricity.

(a) The evolution of the Universe during the symmetric epoch is considered to be predetermined by the coupling of the multiplet of vector fields $U^i_{(a)}$, non-Abelian Yang–Mills field with the potentials $A^{(a)}_k$, and the gravitational field. Objectively, a self-consistent description of this stage is very complicated because of a few reasons. First of all, in order to properly describe the non-Abelian system on this stage we have to use the formalism of the Quantum Field Theory. Second, because some of the vector fields from the set $\{U^i_{(a)}\}$ can be the spacelike or null ones, the theory cannot ignore the appearance of ghosts. Third, generally, the model with spacelike and null vector fields is not stable. The presence of instability complicates the mathematical description of the model, but facilitates the understanding of the idea, that a phase transition is possible in the Universe history, designed to stabilize the situation. In principle, in this stage the idea of Dirac could be developed (many thanks to the first Referee for the hint!). Indeed, because the vector fields $U^i_{(a)}$ are not normalized, one can consider the coincidence (or at least the proportionality) of the gauge field potentials $A^{(a)}_i$ with $U^{(a)}_i$.

(b) The scenario of the transition stage includes three processes: first, the Yang–Mills field becomes quasi-Abelian, $A^{(a)}_i \to Q^{(a)}A_i$, decouples from the interaction with the color vector fields and degrades $A_i \to 0$; second, the procedure of color polarization reduces the color multiplet $U^i_{(a)}$ to one timelike unit vector $U^i_{(a)} \to q_{(a)}U^i$; and third, the color vector $q^{(a)}$ precesses and lines up along the constant color vector $Q^{(a)}$. Of course, such a scenario raises a lot of questions. One of them is connected with the origin of the parallel alignment in the group space of the gauge and vector fields. Another question requires to precise the term spontaneous symmetry breaking. We will try to discuss them below.

(c) The dissymmetric epoch is assumed to be characterized by unit vector field U^i, associated with the canonic aether velocity four-vector; some aspects of the corresponding model are already studied in the framework of isotropic Friedmann type cosmology.

In this paper we focus on the description of the transition stage only. The anisotropic spatially homogeneous Bianchi-I model is considered to be the spacetime platform for three indicated cosmological epochs. We monitor the state of the Universe using the color polarization tensor $H^{(a)(b)}(t)$ as a function of cosmological time. In the symmetric epoch, the rank of the matrix $H^{(a)(b)}$ is assumed to be equal to N^2-1, and the determinant $\mathcal{H} = \det H^{(a)(b)}$ to be non-vanishing. In the dissymmetric epoch, the corresponding rank is equal to one, the determinant is equal to zero. We assume that during the transition epoch N^2-2 eigenvalues of the color polarization tensor (from the total N^2-1 ones) take zero values.

3.1.2. On the Parallel Fields in the Group Space

We assume that the mechanism of the color polarization can be adequately described in terms of fields "parallel" in the group space. This term appeared in the theory of gauge fields for description of the quasi-Abelian gauge field configuration, for which $A^{(a)}_i = Q^{(a)}A_i$ with constants $Q^{(a)}$ satisfying the condition $Q^{(a)}Q_{(a)}=1$ (see the work [62]). Initially, this ansatz was chosen to facilitate the analysis of the Einstein–Yang–Mills equations, and for analysis of colored particle motion (see, e.g., [63,64]. Indeed, for the parallel potentials, all of the nonlinear terms in (7) and in the Yang–Mills equations disappear, thus simplifying the model equations. However, we would like to look at the problem from the other side. Let the gauge field be strong, so that the term $\mathcal{G}f^{(a)}_{(b)(c)}A^{(b)}_m A^{(c)}_n$ makes a significant contribution into the field strength (7). When $A^{(a)}_i = Q^{(a)}A_i$, this term converts into $\mathcal{G}f^{(a)}_{(b)(c)}Q^{(b)}Q^{(c)}A_m A_n \equiv 0$, i.e., it vanishes, even if the potential is huge. Because the energy density of the gauge field contains the square of the $F^{(a)}_{mn}$, one can see that the energy of the non-Abelian configuration has to be bigger than the energy of the one with the parallel fields. From the standard physical point of view, in the process of evolution the system tends to go into the state, for which the energy is less. In other words, the state with parallel potentials may be more preferable for the system and, thus, the vector field alignment is a good mechanism for the minimization of its energy.

Now we apply this ansatz to the vector field assuming that it converts into the product

$$U^i_{(a)} = q_{(a)} U^i \,, \tag{33}$$

but generally the color vector $q^{(a)}$ is not constant and it does not coincide with the quantity $Q_{(a)}$. The gauge covariant derivative can be now transformed into

$$\hat{D}_m U^{(a)}_n = q^{(a)} \nabla_m U_n + U_n \nabla_m q^{(a)} + \mathcal{G} f^{(a)}_{(b)(c)} Q^{(b)} q^{(c)} A_m U_n \,. \tag{34}$$

Clearly, when $q^{(a)}{=}Q^{(a)}$, the potential four-vector A_i drops out from the derivative $\hat{D}_m U^{(a)}_n$, since the structure constant are skew-symmetric. Because the derivative (34) enters the expression of the energy density of the vector field, we can use the same argument: the last term in (34) vanishes, when $q^{(a)}{=}Q^{(a)}$, even if the potential A_i is huge. In other words, in the process of evolution the color vector $q^{(a)}$ has to tend to the color vector $Q^{(a)}$ in order to minimize the total energy of the system.

Keeping in mind the relationship (15) and assuming that the four-vector U^i is unit and time-like

$$g_{ik} U^i U^k = 1 \,, \tag{35}$$

we obtain that the vector $q^{(a)}$ in the group space is also unit

$$G_{(a)(b)} q^{(a)} q^{(b)} = 1 \,. \tag{36}$$

Clearly, the gauge-covariant derivative $\hat{D}_m q^{(a)}$ is orthogonal to the color vector $q_{(a)}$, i.e.,

$$0 = \frac{1}{2} \hat{D}_m [q^{(a)} q_{(a)}] = q_{(a)} \hat{D}_m q^{(a)} = q_{(a)} \left[\nabla_m q^{(a)} {+} \mathcal{G} f^{(a)}_{(b)(c)} A_m Q^{(b)} q^{(c)} \right] {=} q_{(a)} \nabla_m q^{(a)} = 0. \tag{37}$$

The tensor of color polarization $H^{(a)(b)}$ now takes the form

$$H^{(a)(b)} = q^{(a)} q^{(b)} \,. \tag{38}$$

Clearly, the rank of the matrix $H^{(a)}_{(b)}$ is equal to one, and the determinant is equal to zero. The color vector $q^{(a)}$ is an eigenvector of $H^{(a)}_{(b)}$ with the unit eigenvalue:

$$H^{(a)}_{(b)} q^{(b)} = q^{(a)} \,. \tag{39}$$

In other words, the mechanism of parallelization in the group space applied to the color vectors $U^i_{(a)}$ gives the result that is analogous to the polarization of light in optics. Indeed, one can say that a special direction in the group space appears, which is associated with the color vector $q^{(a)}$, along which all of the color vectors $U^i_{(a)}$ are lined up. Generally, $q^{(a)}(t)$ precesses in the group space, but when the color vector $q^{(a)}$ becomes constant, one can transform the matrix $H^{(a)(b)}$ into diag$\{1, 0, 0...0\}$ for arbitrary time moment, using the admissible rotation in the group space. The description of the evolution of the color vector $q^{(a)}(t)$ is the crucial point of our analysis.

3.1.3. On the Spacetime Platform

For the illustration of the suggested idea, we consider the class of homogeneous spacetimes with the metric

$$ds^2 = dt^2 - a^2(t) dx^2 - b^2(t) dy^2 - c^2(t) dz^2 \,. \tag{40}$$

Generally, this metric describes the Bianchi-I model. When $a(t){=}b(t){=}c(t)$, we obtain the Friedmann type model. We assume that all of the unknown model state functions inherit the spacetime symmetry and depend on the cosmological time only.

The global unit timelike vector U^i is assumed to be of the form $U^i = \delta^i_t$; below, we show that the model admits this construction. The metric (40) provides the covariant derivative to be symmetric, the acceleration four-vector a_i to vanish, and the scalar of expansion Θ to have very simple form

$$\nabla_m U_n = \nabla_n U_m = \frac{1}{2}\dot{g}_{mn} \,, \quad a_i \equiv U^k \nabla_k U_i = 0 \,, \quad \Theta(t) \equiv \nabla_k U^k = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \,. \tag{41}$$

Here and below the dot denotes the ordinary derivative with respect to time. In addition, we use two differential consequences of the normalization conditions: $U^k \nabla_i U_k = 0$ and $q_{(c)} U^k \nabla_k q^{(c)} = 0$. The listed assumptions simplify the model essentially, and we start to discuss its details.

3.2. Reduced Master Equations

3.2.1. The Yang–Mills Field in the Transition Epoch

As we already noticed, we assume that the first manifestation of the phase transition in the model under discussion is connected with the parallelization of the Yang–Mills potentials, i.e., $A^{(a)}_i \to Q^{(a)} A_i$, where $Q^{(a)}$ is a constant vector in the group space; the second manifestation is the parallelization of the color vector fields $U^{(a)}_i \to q^{(a)} U_i$. For such fields, parallel in the group space, the strength of the Yang–Mills field (7) has the quasi-Maxwellian form

$$F^{(a)}_{mn} = Q^{(a)} [\nabla_m A_n - \nabla_n A_m] \,. \tag{42}$$

Based on the Bianchi-I spacetime platform and taking into account the antisymmetric properties of the structure constants, we find the color current $\Gamma^i_{(a)}$ that is produced by the color vector fields:

$$\Gamma^i_{(a)} = \frac{\mathcal{G}}{\kappa}(C_1{+}C_2{+}C_3) U^i f_{(d)(c)(a)} q^{(c)} \left[\dot{q}^{(d)} - \mathcal{G} f_{(d)(b)(h)} Q^{(b)} q^{(h)} U^m A_m \right] -$$
$$- \frac{\mathcal{G}^2}{\kappa} C_1 A^k \Delta^i_k f^{(d)}_{(c)(a)} q^{(c)} f_{(d)(b)(h)} Q^{(b)} q^{(h)} . \tag{43}$$

Here, $\Delta^i_k \equiv \left(\delta^i_k - U^i U_k\right)$ is the projector. Clearly, as long as the color current is not vanishing, $\Gamma^i_{(a)} \neq 0$, or is not linear in the potential A_i, the Yang–Mills Equation (18) do not admit the trivial solutions $A_i = 0$. In principle, this color current disappears, when $q_{(a)} = Q_{(a)} = const$; we consider this state as the final one. When $q^{(a)}$ depends on time and, thus, $\dot{q}^{(a)} \neq 0$, one can obtain the trivial solution $A_i = 0$, if to use the special condition for the Jacobson's coupling constants

$$C_1 + C_2 + C_3 = 0 \,, \tag{44}$$

which was motivated, e.g., in [54]. For this case, the color current is proportional to the potential A^k, and the trivial exact solution $A^i = 0$ exists. This means that the system can choose this trivial solution for the Yang–Mills field in the bifurcation associated with the end of the first phase transition. Of course, the condition (44) narrows down the frameworks of the canonic Einstein–aether theory; however, this restricted model is very illustrative for our purpose. By the way, if we accept that $C_1 + C_3 \equiv 0$ (see the subsection 1.1. for motivation), the condition (44) gives $C_2 \equiv 0$, i.e., $S^2_{(0)} = 0$ and the spin-0 aether modes can not propagate.

3.2.2. Solutions to the Equations for the Color Vector Fields

Now we can consider the reduced master equations for the multiplet of vector fields with the conditions $A^i_{(a)}{=}0$ and $C_1{+}C_2{+}C_3{=}0$. Clearly, the vectorial quantity $\mathcal{I}^j_{(a)}$ (22) vanishes, since the following vector is equal to zero:

$$U^n_{(b)}\hat{D}_n U^{m(b)}{=}q_{(b)}q^{(b)}U^n\nabla_n U^m{+}U^m q_{(b)}\dot{q}^{(b)}. \tag{45}$$

The tensor $\mathcal{J}^{ij}_{(a)}$ (21) happens to be proportional to the coupling constant C_2:

$$\mathcal{J}^{ij}_{(a)}{=}C_2\left\{q_{(a)}\left[g^{ij}\Theta{-}\nabla^{(i}U^{j)}\right]{+}\dot{q}_{(a)}\Delta^{ij}\right\}. \tag{46}$$

Subsequently, we see that $3(N^2{-}1)$ equations from the $4(N^2{-}1)$ ones, written in (20) convert into the trivial identities $0=0$; only $N^2{-}1$ equations for $j=0$ have the nontrivial form

$$q_{(a)}C_2\left(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}\right)=\lambda q_{(a)}\,. \tag{47}$$

All of these equations are satisfied simultaneously, if the Lagrange parameter $\lambda(t)$ is chosen to be of the form:

$$\lambda=C_2\left(\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}\right)\,. \tag{48}$$

We have to emphasize that, in the presented model, the master equations for the multiplet of vector fields do not impose any restrictions on the color vector $q^{(a)}(t)$. This means that we can suggest our version of the model of the color vector $q^{(a)}(t)$ evolution.

3.2.3. Gravity Field Equations

To explain the details of the structure of the reduced aether stress-energy tensor (25) we have to make four preliminary remarks.

1. Using (46) and the relationships

$$\nabla_i(q^{(a)}U_j)=\frac{1}{2}q^{(a)}\dot{g}_{ij}{+}\dot{q}^a U_i U_j \tag{49}$$

we can calculate the scalar

$$\mathcal{K}^{ijmn}_{(a)(b)}\hat{D}_i U^{(a)}_m\hat{D}_j U^{(b)}_n{=}C_2\left[\Theta^2{-}\nabla_i U_j\nabla^i U^j\right]=2C_2\left[\frac{\dot{a}}{a}\frac{\dot{b}}{b}{+}\frac{\dot{a}}{a}\frac{\dot{c}}{c}{+}\frac{\dot{b}}{b}\frac{\dot{c}}{c}\right]. \tag{50}$$

2. Because the tensor $\nabla_i U_j$ is symmetric, the term proportional to C_1 in (25) vanishes.
3. Because $U^k\nabla_k U_j=0$ and $q_{(a)}\dot{q}^{(a)}=0$, the term with C_4 in (25) also vanishes.
4. Because $\mathcal{J}^{(a)}_{ij}=\mathcal{J}^{(a)}_{ji}$, the last term in (25) converts into

$$C_2\left[-g_{pq}\left(\dot{\Theta}+\Theta^2\right){+}\frac{1}{2}\Theta\dot{g}_{pq}{+}U^m\nabla_m(\nabla_p U_q)\right]. \tag{51}$$

Thus, for the tensor T^p_q we obtain the formula, which surprisingly contains neither the color vector $q^{(a)}$ nor its derivative $\dot{q}^{(a)}$:

$$\begin{gathered}T^p_q{=}C_2\left\{-\delta^p_q\left(\frac{\ddot{a}}{a}{+}\frac{\ddot{b}}{b}{+}\frac{\ddot{c}}{c}{+}\frac{\dot{a}}{a}\frac{\dot{b}}{b}{+}\frac{\dot{a}}{a}\frac{\dot{c}}{c}{+}\frac{\dot{b}}{b}\frac{\dot{c}}{c}\right)\right.\\ \left.{+}\delta^p_1\delta^1_q\left(\frac{\ddot{a}}{a}{+}\frac{\dot{a}}{a}\frac{\dot{b}}{b}{+}\frac{\dot{a}}{a}\frac{\dot{c}}{c}\right){+}\delta^p_2\delta^2_q\left(\frac{\ddot{b}}{b}{+}\frac{\dot{b}}{b}\frac{\dot{c}}{c}{+}\frac{\dot{a}}{a}\frac{\dot{b}}{b}\right){+}\delta^p_3\delta^3_q\left(\frac{\ddot{c}}{c}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}\right)\right\}\,.\end{gathered} \tag{52}$$

Taking into account the term for λ (48), we can now represent four non-trivial gravity field equations in the following form:

$$\left[\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}\right](1+C_2)=\Lambda\,, \quad \left[\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}\right](1+C_2)=\Lambda\,,$$
$$\left[\frac{\ddot{a}}{a}+\frac{\ddot{c}}{c}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}\right](1+C_2)=\Lambda\,, \quad \left[\frac{\ddot{b}}{b}+\frac{\ddot{a}}{a}+\frac{\dot{b}}{b}\frac{\dot{a}}{a}\right](1+C_2)=\Lambda\,. \tag{53}$$

It is the well known system of equations, the novelty is only in the structure of the constant: we deal now with the modified cosmological constant $\frac{\Lambda}{1+C_2}$ instead of Λ. The symmetry of the Equation (53) allows us to explicitly find the expansion scalar $\Theta(t)$ as the function of time. Indeed, if we calculate the auxiliary function $\dot{\Theta}+\Theta^2$, we obtain

$$\dot{\Theta}+\Theta^2=\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}+2\left[\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}\right]. \tag{54}$$

The appropriate linear combination of the Equation (53) shows that the right-hand side of (54) is equal to $\frac{3\Lambda}{1+C_2}$. In other words, we are faced with the equation

$$\dot{\Theta}+\Theta^2=\frac{3\Lambda}{1+C_2}\,, \tag{55}$$

the solution to which is the following:

$$\Theta(t)=3H_0\left\{\frac{\Theta(t_0)+3H_0\,\mathrm{th}[3H_0(t-t_0)]}{\Theta(t_0)\,\mathrm{th}[3H_0(t-t_0)]+3H_0}\right\}, \tag{56}$$

where the parameter H_0 is introduced as

$$H_0\equiv\sqrt{\frac{\Lambda}{3(1+C_2)}}\,. \tag{57}$$

The function $\Theta(t)$ starts from the value $\Theta(t_0)$ at $t=t_0$ and finishes with the value $3H_0$ at $t\to\infty$. When $|\Theta(t_0)|<3H_0$, the function $\Theta(t)$ is regular in the interval $t_0<t<\infty$, and it is monotonic with $\dot{\Theta}>0$. When $\Theta(t_0)=3H_0$, the solution is constant, $\Theta(t)=3H_0$.

Additionally, one can find the explicit formula for the unit volume in the three-dimensional space

$$V(t)=\frac{a(t)\cdot b(t)\cdot c(t)}{a(t_0)b(t_0)c(t_0)}\,, \tag{58}$$

for which $V(t_0)=1$, and

$$\frac{\dot{V}}{V}=\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}=\Theta(t)=\nabla_k U^k\,. \tag{59}$$

Using (56) and (59), we now obtain

$$V(t)=\frac{\Theta(t_0)}{3H_0}\mathrm{sh}[3H_0(t-t_0)]+\mathrm{ch}[3H_0(t-t_0)]\,. \tag{60}$$

Finally, it is clear that the differences of logarithmic derivatives have the form

$$\frac{\dot{b}}{b}-\frac{\dot{a}}{a}=\frac{K_1}{V(t)}, \quad \frac{\dot{c}}{c}-\frac{\dot{b}}{b}=\frac{K_2}{V(t)}, \quad \frac{\dot{a}}{a}-\frac{\dot{c}}{c}=\frac{K_3}{V(t)}, \tag{61}$$

where K_1, K_2, K_3 are the integration constants, which satisfy the condition $K_1{+}K_2{+}K_3{=}0$. To make sure that (61) take place, we can check, e.g., due to the second and third equations from the set (53)

$$\frac{d}{dt}\left(\frac{\dot{b}}{b}-\frac{\dot{a}}{a}\right) = -\left(\frac{\dot{b}}{b}-\frac{\dot{a}}{a}\right)\Theta\,, \tag{62}$$

and then integrate this equation. Asymptotically, at $V \to \infty$, the rates of expansion $\frac{\dot{a}}{a}$, $\frac{\dot{b}}{b}$ and $\frac{\dot{c}}{c}$ coincide, i.e., the Universe tends to be spatially isotropic. If we are interested to find the functions $a(t)$, $b(t)$, $c(t)$, we can use the relationships

$$\frac{\dot{a}}{a} = \frac{1}{3}\Theta + \frac{K_3-K_1}{3V}\,, \quad \frac{\dot{b}}{b} = \frac{1}{3}\Theta + \frac{K_1-K_2}{3V}\,, \quad \frac{\dot{c}}{c} = \frac{1}{3}\Theta + \frac{K_2-K_3}{3V}\,, \tag{63}$$

supplemented by (56) and (60), and then integrate them. Finally, we can state the following.

(i) When the cosmological constant is equal to zero, $\Lambda = 0$, and $C_2 \neq -1$, the Equation (53) give the known anisotropic Kasner solutions, and the parameter C_2, associated with the aether, becomes hidden.
(ii) When $\Lambda \neq 0$, we deal with the spacetime isotropization, and we asymptotically obtain the de Sitter spacetime with $a(t) \propto \exp\left\{\sqrt{\frac{\Lambda}{3(1+C_2)}}\,t\right\}$; in this case the coupling constant C_2 modifies the rate of expansion.
(iii) When $\Lambda = 0$ and $C_2 = -1$ simultaneously, the gravity field Equation (53) are satisfied identically, so that the metric functions $a(t)$, $b(t)$ and $c(t)$ are arbitrary (in the presented toy model).
(iv) When $C_3 = -C_1$ and $C_2 = 0$, the parameters C_1 and C_4 happen to be hidden, and there are no imprints of the dynamic aether in the equations for the gravity field describing the Universe evolution (dynamic aether becomes invisible?).

3.3. *Dynamics of the Color Vector $q^{(a)}(t)$*

In the scenario proposed above, the minimization of the energy of the gauge field can be reached, if $A^{(a)}_i = Q^{(a)}A_i$; the minimization of the interaction energy between the aether and the gauge field requires the color vector $q^{(a)}$ to coincide with $Q^{(a)}$. However, during the first stage of the phase transition, we deal with the relationship $q^{(a)} \neq Q^{(a)}$, and one needs to describe the mechanism of re-orientation of the color vector $q^{(a)}$. In this subsection we propose our version of the color vector evolution; we indicate it as the precession mechanism.

3.3.1. Exact Solution to the Precession Equation

According to our scenario, after the phase transition, when the specific direction $Q^{(a)}$ in the group space appeared, and the color vector fields have become parallel, $U^i_{(a)}{=}q_{(a)}U^i$, the process of precession of the color vector $q^{(a)}$ around the color vector $Q^{(a)}$ was started up. This process can be described by the equation, which is the generalization of the Wong equation for the color charges [65]

$$U^m \hat{D}_m q^{(a)} = \mathcal{F}^{(a)}\,, \quad \mathcal{F}^{(a)} = \nu\left[\delta^{(a)}_{(b)} - q^{(a)}q_{(b)}\right]Q^{(b)}\,. \tag{64}$$

In the Bianchi-I spacetime platform, these equations can be presented in more detailed form:

$$\dot{q}^{(a)} + \mathcal{G}f^{(a)}_{(b)(c)}Q^{(b)}q^{(c)}(U^m A_m) = \nu(t)\left[Q^{(a)} - q^{(a)}q_{(b)}Q^{(b)}\right]\,, \tag{65}$$

where $\nu(t)$ is some function of time. The force-like term $\mathcal{F}^{(a)}$ in (64) is orthogonal to the color vector $q_{(a)}$: i.e., $q_{(a)}\mathcal{F}^{(a)}{=}0$. This property of the force-like term $\mathcal{F}^{(a)}$ guaranties that the normalization condition $q_{(a)}q^{(a)} = 1$ is supported. The second term in the left-hand side of the Equation (65) is also orthogonal to the color vector $q_{(a)}$; it can be eliminated using the so-called Landau's gauge condition for the Abelian potential four-vector, $U^i A_i = 0$. The equations, modified correspondingly,

$$\dot{q}^{(a)} = \nu(t) \left[Q^{(a)} - q^{(a)} q_{(b)} Q^{(b)} \right] , \tag{66}$$

look like the equations of motion of the macroscopic particle with unit mass under the influence of the Stokes-type force in the fluid flow, which is characterized by the velocity $Q^{(a)}$. In order to solve the Equation (66), we consider the scalar product $X(t) \equiv q_{(a)}(t) Q^{(a)}$; since $q^{(a)}(t) q_{(a)}(t) = 1$ and $Q^{(a)} Q_{(a)} = 1$, the scalar product X is equal to the cosine of the angle between the vectors $q^{(a)}$ and $Q^{(b)}$ in the group space. Then we find its derivative $\dot{X} = Q_{(a)} \mathcal{F}^{(a)}$, and obtain the equation, which is very famous in the theory of differential equations:

$$\dot{X} = \nu(t)(1 - X^2) . \tag{67}$$

This equation admits two special constant solutions $X{=}{\pm}1$, which relate to the cases $q^{(a)}(t){=}{\pm}Q^{(a)}$, i.e., $q^{(a)}$ and $Q^{(a)}$ are parallel or anti-parallel, respectively, for arbitrary moment of the cosmological time. For arbitrary initial value $X(t_0)$, the solution to the Equation (67) is known to be of the following form:

$$X(t) = \frac{e^{2T(t)}(1 + X(t_0)) - (1 - X(t_0))}{e^{2T(t)}(1 + X(t_0)) + (1 - X(t_0))} , \quad T(t) \equiv \int_{t_0}^{t} d\tau \nu(\tau) . \tag{68}$$

Returning to the Equation (66) rewritten in the form

$$\frac{dq^{(a)}}{dT} = Q^{(a)} - q^{(a)} X(T) , \tag{69}$$

we readily find the solution

$$q^{(a)}(t) = \frac{q^{(a)}(t_0)}{[\text{ch}T + X(t_0)\text{sh}T]} + Q^{(a)} \frac{[\text{sh}T + X(t_0)\,(\text{ch}T - 1)]}{[\text{ch}T + X(t_0)\text{sh}T]} . \tag{70}$$

For illustration, it is convenient to consider the case with $X(t_0) = 0$; it corresponds to the configuration with initially orthogonal color vectors $q^{(a)}(t_0)$ and $Q^{(a)}$; now we obtain

$$X(t) = \text{th}T , \quad q^{(a)}(t) = \frac{q^{(a)}(t_0)}{\text{ch}T} + Q^{(a)} \text{th}T , \tag{71}$$

$$H^{(a)(b)}(t) = \frac{H^{(a)(b)}(t_0)}{\text{ch}^2 T} + Q^{(a)} Q^{(b)} \text{th}^2 T + \left[Q^{(a)} q^{(b)}(t_0) + Q^{(b)} q^{(a)}(t_0) \right] \frac{\text{sh}T}{\text{ch}^2 T} . \tag{72}$$

Asymptotically, at $T \to \infty$ we obtain $q^{(a)} \to Q^{(a)}$. In other words, during the evolution process, the color vector $q^{(a)}$ forgets its initial value $q^{(a)}(t_0)$ and it is forced to be lined up along the direction $Q^{(a)}$. The question arises: what is the rate of asymptotic parallelization of the vectors $q^{(a)}$ and $Q^{(d)}$. Clearly, this process is predetermined by the properties of the function $\nu(t)$; let us consider an example of its modeling.

3.3.2. Model Function $\nu(t)$

For the modeling of the function $\nu(t)$, we use the following construction:

$$\nu(t) = 2\gamma T_* \Theta(t) \frac{V^2}{[V_*^2 - V^2(t)]^{1+\gamma}} , \tag{73}$$

and calculate the function $T(t)$ (68):

$$T(t) = T_* \left\{ \frac{1}{[V_*^2 - V^2(t)]^\gamma} - \frac{1}{[V_*^2 - 1]^\gamma} \right\} . \tag{74}$$

Here T_* and γ are positive parameters. Let us make three remarks concerning this formula.

1. When the function $V(t)$ grows, it reaches the value V_*, so, at the finite moment of the cosmological time $t = t_*$ the function $T(t)$ reaches infinity, $T(t_*) = \infty$.
2. According to (70), at $T(t_*) = \infty$, the color vector $q^{(a)}(t_*)$ coincides with $Q^{(a)}$, i.e., the process of their parallelization finished during the finite time interval.
3. At the moment $t = t_*$, the derivative of the color vector $q^{(a)}$ takes zero value, $\dot{q}^{(a)}(t_*) = 0$.

In order to illustrate the last statement, we calculate the derivative of the function $q^{(a)}(t)$ presented in (71) for the case $X(t_0) = 0$:

$$\dot{q}^{(a)}(t) = \frac{\nu(t)}{\mathrm{ch}T} \left[\frac{Q^{(a)}}{\mathrm{ch}T} - q^{(a)}(t_0)\mathrm{th}T \right] . \tag{75}$$

For big values of the function T, one can use the following replacement (compare (73) and (74)):

$$\nu(t) \to 2\gamma T_*^{-\frac{1}{\gamma}} \Theta V^2(t_*) T^{1+\frac{1}{\gamma}} , \tag{76}$$

where Θ (56) is the regular finite function. Because, for arbitrary α, the limit $\lim_{T\to\infty} \left[\frac{T^\alpha}{\mathrm{ch}T} \right]$ is equal to zero, we can confirm the condition $\dot{q}^{(a)}(t_*) = 0$. The critical time moment t_*, for which $V(t_*) = V_*$ can be found as

$$t_* = t_0 + \frac{1}{3H_0} \ln \left\{ \frac{3H_0 V_*}{3H_0 + \Theta(t_0)} \left[1 + \sqrt{1 + \frac{\Theta^2(t_0) - 9H_0^2}{9H_0^2 V_*^2}} \right] \right\} . \tag{77}$$

One can check that $t_* > t_0$, when $V_* > 1$. For the special case, when $3H_0 = \Theta(t_0) = \Theta(t)$, we obtain $t_* = t_0 + \frac{1}{3H_0} \ln V_*$.

3.3.3. Model Threshold Function Ω

Let us now consider the function Ω introduced by (13) and (14), the value of which indicates the nearness of the event of the spontaneous color polarization. We have now that

$$\Omega^{(a)} = \dot{q}^{(a)} + q^{(a)}\Theta , \quad \Omega^2 = \dot{q}^{(a)}\dot{q}_{(a)} + \Theta^2 , \tag{78}$$

thus, for the illustrative function (71) we obtain

$$\Omega = \sqrt{\frac{\nu^2}{\mathrm{ch}^2 T} + \Theta^2} . \tag{79}$$

Taking into account (73) and (74), we finally reconstruct the function Ω as

$$\Omega = \Theta \sqrt{1 + \frac{4\gamma^2 T_*^2 V^4}{\mathrm{ch}^2 T} \left[\frac{T}{T_*} + \frac{1}{(V_*^2 - 1)^\gamma} \right]^{\frac{2(1+\gamma)}{\gamma}}} . \tag{80}$$

Near the critical point $t = t_*$, when $T \to \infty$, the threshold function Ω behaves as the expansion scalar Θ, i.e., the summarized rate of the Universe expansion $\Theta = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}$ predetermines the rate of the onset of the spontaneous color polarization.

4. Discussion and Conclusions

We share the point of view of many cosmologists, that a plethora of fields and particles in the early hot Universe were lost, destroyed, or transformed as a result of a series of phase transitions; now, we observe a limited number of fundamental fields and stable elementary particles. Our goal is to reconstruct the history of emergence of the cosmic substratum indicated as dynamic aether; to be more precise, we are interested to establish a model of formation of the unit timelike vector field as a splinter of the family of vector fields from the SU(N) symmetric multiplet. Our phenomenological model is based on three elements.

(1) The first element of the model is the procedure of the spontaneous polarization of the SU(N) multiplet of vector fields. This procedure is described in terms of critical behavior of the eigenvalues of the color polarization tensor (see (27), (31) and (32)). When the Universe expands, and the scalar Ω reaches the critical value $\Omega^*_{\{\alpha\}}$, the corresponding eigenvalue with the serial number α vanishes. When $\Omega > \max\left\{\Omega^*_{\{\alpha\}}\right\}$, we obtain the situation with only one non-vanishing eigenvalue; this last eigenvalue is equal to one, because the trace of the polarization tensor is unit. Based on the analogy with optics, we have to state now, that we deal with linearly polarized system, i.e., the color polarization took place. Mathematically, this procedure means that all of the vectors from the SU(N) symmetric multiplet have become parallel in the group space to one vector U^i, namely, $U^i_{(a)} = q_{(a)}U^i$.

(2) The second element of the model is the mechanism of precession of the color vector $q^{(a)}$ around the constant color vector $Q^{(a)}$ formed in course of parallelization of the Yang–Mills potential $A^{(a)}_i \to Q^{(a)}A_i$. Evolution of the color vector $q^{(a)}$ was described by the equation of the Wong type (64); the exact solution to this equation is presented in (70). Asymptotically, $q^{(a)}(t)$ tends to the constant $Q^{(a)}$, and the system as a whole becomes quasi-Abelian, since all of the structure constants $f_{(a)(b)(c)}$ fall out from the basic formulas.

(3) The third element of the model is the calculation of the rate of the precession of the color vector $q^{(a)}$. We have chosen the guiding function $\nu(t)$ in (65) in the critical form (73) providing the process of parallelization of $q^{(a)}$ and $Q^{(a)}$ to be finished during the finite time interval (see (70) and (74)).

In the context of our study, we can not elude the question concerning the breaking of a symmetry; what is the symmetry, which we mean? In our scenario we assume that a phase transition of the second kind takes place in the early Universe. According to the Landau's theory of the phase transitions in crystals, one can indicate explicitly the spatial symmetry groups, which characterize the system before and after the transition, and one can point out a specific order parameter (the scalar, spatial vector, or tensor). In our case, the spacetime symmetry in the Universe is not broken, and the restructuring takes place in the field conglomerate and concerns the internal symmetry of the self-interacting system. Before such an effective phase transition the set of interacting fields could be characterized by the SU(N) symmetry group, and after the transition these fields display the symmetry, which relates to the U(1) group of internal symmetry. In other words, the group space is the scene of the transition; the transformations $A^{(a)}_i \to Q^{(a)}A_i$ and $U^{(a)}_i \to q^{(a)}U_i$ relate to first act of the spectacle, as a result the group space becomes bi-axial, because two color vectors do not coincide, $q^{(a)} \neq Q^{(a)}$. During the second act, the color vector $q^{(a)}$ precesses and tends to be equal to $Q^{(a)}$; at the final of the second act the group space becomes uni-axial; the corresponding color vector plays the role of the order parameter of this effective phase transition. What is the driven force of this phase transition? We think that it is the natural tendency of the physical system to find the state with minimal energy level; since the energy of the quasi-Abelian configuration is less than the energy of the non-Abelian one, the transition is inevitable.

The conclusion of our study is the following: the aether velocity four-vector can be considered as a rudiment, which remained after the spontaneous color polarization accompanying the phase transition in the early Universe.

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