Models of nonlinear wave processes that allow for soliton solutions

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A general method for identifying models that allow for soliton solutions from general classes of models of nonlinear wave processes with fixed dispersion laws is examined. The method uses the generalized Lagrange identity for adjoint equations and is universal in relation to the type of dispersion of the initial class of equations and their vector dimensionality. This makes it possible to build generalized versions of known and new exactly solvable equations such as nonlinear Schrödinger equations, Korteweg–de Vries equations, etc., by specifying explicitly the type of the L–A pair for each equation, which makes it possible to immediately apply the inverse scattering method. The new method can also be used to build generalized versions of models of the interaction of N waves in media with a quadratic dispersion law. © *1996 American Institute of Physics.* [S1063-7761(96)02512-7]

1. INTRODUCTION

With the arrival of the inverse scattering method physics acquired an entirely new class of universal exactly solvable models describing, from a unified standpoint, nonlinear wave processes in various dispersive media in conditions when nonlinearity and dispersion are balanced. Among the models are those based on Korteweg–de Vries equations, nonlinear Schrödinger equations, modified Korteweg–de Vries equations, the three-wave interaction, etc.[1] In addition to exact integrability, the importance of these models is determined by the fact that the nonlinear wave processes (solitons, kinks, breezers, and the like) described by the models are usually stable under external disturbances, which is important because of the possibility of observing them in real physical systems. Although this class of equations was found to be fairly limited, within the standard procedure of multiscale expansions in the theory of perturbations with exclusion of resonances these models appear regularly in various, often unrelated, sections of physics, which is apparently a consequence of an implicit requirement that the solutions be stable, a requirement imposed by the procedure of solving equations via perturbation theory with exclusion of resonances.

At the same time, equations describing the real problems in constructing these exactly integrable models often acquire additional terms, which can disrupt the balance of nonlinearity and dispersion. This leads to rapid loss of the stability of nonlinear waves of the soliton type. Hence in such cases it is desirable to know under what additional conditions an integrable model with additional terms still allows for exact solutions of the soliton type.

Often in applied problems (see, e.g., Refs. 2 and 3), to establish how the dynamics of the solitons is influenced by the medium's parameters the researcher is forced to examine the evolution of the one-soliton solution under a perturbation. Usually the problem is solved approximately by perturbation-theory techniques. But the perturbation theory of solitons (see, e.g., Refs. 4 and 5) does not answer the important question is whether the equation in question has soliton solutions, i.e., sets of solitary waves interacting elastically. These properties guarantee that slow variations of the properties of one solitary wave are the same as in the set of waves.

The problems whose solution answers this question may be formulated in different ways, one of which reduces to calculating the general form of the evolutionary equations of the model that allow for soliton solutions for a giver order of the medium's dispersion and a given number of waves participating in a weakly nonlinear wave process. Different approaches have been used in the study of this problem and interesting results have been obtained. In Refs. 1,6–8 a way of calculating the equations that can be integrated by the inverse scattering method was proposed. It was based on the procedure of "dressing" the operators of the Lax representation. Other methods for calculating the form of the equations, methods related to the symmetry algebra of differential equation, have also been proposed (see, e.g., Refs. 9,10, and 11). The Wahlquist–Estabrook approach[10,12] also belongs to this class of models. The main difficulty in using these methods is that the form of the equations obtained with their help is not related from the start to the form of the initial equation but is determined by the procedure of building such equations or the type of symmetry. However, these methods make it immediately possible to apply the inverse scattering method to the emerging equations, since they provide a pair of operators for the Lax representations for each equation.

Another approach is related to a direct verification of the fact that the equations have a rich set of conservation laws, which is the Liouville definition of complete integrability of equations. In Refs. 13 and 14 this problem was solved by proving the existence of solutions of operator equations of a certain type related to the initial equation. This makes it possible to build lists of integrable equations (see also Ref. 15). This approach, however, does not generally allow calculating the Lax operators for each equation, which hinders its practical use.

Another approach to the solution of the problem is based on "deformations" of the Lax representations, which actually means performing substitutions of a general form of the coordinate variables and the spectral parameter in the operators of this representation (see, e.g., Refs. 16 and 17). The approach requires using from the start an equation with a known Lax pair of operators and makes it possible to establish new classes of equations that can be integrated by the inverse scattering method and are obtained as a result of such deformations.

The problem of finding the form of equations that allow for soliton solutions was solved in Ref. 18 for the three-wave interaction model by a method based on the Lagrange identity. The main idea of using the Lagrange identity for solving the problem is that the Lagrange identity provides a simple and universal way of building all the conservation laws for any system described by finite-order partial differential equations. To establish when the problem is completely or partially integrable we must extract additional information about the structure of the laws, i.e., we must check whether they are in involution or not. In Ref. 18 it was shown that the Lax representation is a direct consequence of the Lagrange identity. An appropriate interpretation of this fact immediately provides a way of formally building such representations (even in the case where the equations are not integrable). The problem that follows consists in finding the conditions under which the obtained representation is exactly of the Lax type, i.e., contains a spectral parameter in an explicit and nontrivial manner. Actually this approach makes it possible to obtain all the equations in the general class of initial equations of $1+1$ dimensionality that allow for soliton solutions together with all admissible deformations of such equations. In contrast to the approaches and methods mentioned earlier, this approach provides a universal scheme for calculating the equations that is independent of the type of dispersion and the vector dimensionality of the initial equation.

The present paper studies the application of this method to solving the general problem of the existence of soliton waves in media with quadratic, cubic, etc. dispersion and propagating in a one-dimensional nonlinear medium. First the method and its main elements are illustrated by an example of solitary waves propagating in media with quadratic and cubic dispersion, with the nonlinear Schrödinger and Korteweg–de Vries equations being typical examples of the corresponding equations. The equations of this type are well-known, but not all types of deformations given here have been described before. Vector and multicomponent equations[19,20] comprise a poorly studied area of the theory of equations that admit for soliton solutions. The aim of the present work is to apply the method to problems in which a finite number of waves interact in media with quadratic dispersion, with the three-wave interaction used as an example. The problem is important from the practical standpoint for analyzing the propagation of short and ultrashort optical pulses in nonlinear optical media.[21,22]

2. THE LAGRANGE IDENTITY FOR THE EQUATION OF A SINGLE WAVE PROPAGATING IN A MEDIUM WITH QUADRATIC DISPERSION

Let us take as an example the model of a single wave propagating in a one-dimensional nonlinear medium with quadratic dispersion. The wave process of this type is represented by the real-valued function

$$\mathscr{E}(x,t)=u(x,t)e^{i(kx-\omega t)}+u^*(x,t)e^{-i(kx-\omega t)},$$

which describes the deflection of the medium (or the field in the medium) from equilibrium as the wave passes through a point with a coordinate x at time t. Here $u(x,t)$ is the wave's complex-valued amplitude, and k and ω are the wave number and frequency. The general equation describing the slow variation of the function $u(x,t)$ in a medium with quadratic dispersion has the form

$$\varepsilon u_t+r_2(x,t;u)u_{xx}+r_1(x,t;u)u_x+r_0(x,t;u)u=0. \tag{1}$$

Here $\varepsilon=\text{const}$, and the following notation has been introduced:

$$u_t=\frac{\partial u}{\partial t},\quad u_x=\frac{\partial u}{\partial x},\quad u_{xx}=\frac{\partial^2 u}{\partial x^2},\quad \dots .$$

The coefficients $r_i(x,t;u)$, $i=0,1,2$, considered as functions of x and t describe the variation of the medium, while their dependence on the functions $u(x,t)$ describes the nonlinear properties of the medium. An important problem that emerges in the study of such models is finding the form of the coefficients $r_i(x,t;u)$ for which Eq. (1) admits for multisoliton solutions.

By analogy with linear differential equations, we can introduce for Eq. (1) the concepts of an adjoint function and an adjoint equation. The general theory of linear differential operators states that for a linear operator $\mathbf{L}$ acting on functions $u(x,t)$ the action of the adjoint operator $\bar{\mathbf{L}}$ is defined in such a way that for any functions $\phi(x,t)$ and $u(x,t)$ in a domain $\Omega\subset R^2$ of variation of the arguments x and t we have

$$\int_\Omega \phi(\mathbf{L}u)\,dx\,dt-\int_\Omega(\bar{\mathbf{L}}\phi)u\,dx\,dt$$
$$=\int_{\partial\Omega}Q_1(x,t)\,dx+Q_0(x,t)\,dt,$$

where $\mathbf{Q}=(Q_0,Q_1)$ is a vector field on Ω whose form is determined entirely by the operators $\mathbf{L}$ and $\mathbf{L}'$ and can be calculated by using the generalized Lagrange identity

$$\phi(\mathbf{L}u)-(\bar{\mathbf{L}}\phi)u\equiv\frac{\partial}{\partial t}Q_0(x,t)+\frac{\partial}{\partial x}Q_1(x,t). \tag{2}$$

Assuming that for Eq. (1)

$$\mathbf{L}=\varepsilon\frac{\partial}{\partial t}+r_2(x,t)\frac{\partial^2}{\partial x^2}+r_1(x,t)\frac{\partial}{\partial x}+r_0(x,t), \tag{3}$$

we can find the adjoint operator:

$$\bar{\mathbf{L}}=-\varepsilon\frac{\partial}{\partial t}+\frac{\partial^2}{\partial x^2}r_2(x,t)-\frac{\partial}{\partial x}r_1(x,t)+r_0(x,t). \tag{4}$$

In the above equations the explicit dependence of r_0, r_1, and r_2 on the unknown function $u(x,t)$ has been dropped for the time being, i.e., it is assumed that the dependence on x and t takes into account the possible dependence on $u(x,t)$. The components of the vector field $\mathbf{Q}(x,t)$ corresponding to (2) are

$$Q_1(x,t)=r_1u\phi+u_x\phi r_2-u(\phi r_2)_x,\quad Q_0(x,t)=\varepsilon u\phi.$$

Let $u(x,t)$ and $\phi(x,t)$ be the solutions of the respective equations,

$$\mathbf{L}u(x,t)=0,\quad \bar{\mathbf{L}}\phi(x,t)=0. \tag{5}$$

Then, according the Lagrange identity (2), the following generalized differential conservation law holds:

$$\phi(\mathbf{L}u)-(\bar{\mathbf{L}}\phi)u\equiv\frac{\partial}{\partial t}Q_0(x,t)+\frac{\partial}{\partial x}Q_1(x,t)=0. \tag{6}$$

If the function $u(x,t)$ is fixed, $\phi(x,t)$ satisfies a linear equation and therefore is formally a function of a single spectral parameter k. Consequently, Eq. (6) contains not a single conservation law but a set of such laws, which are the coefficients of the series expansions of the components of the vector field $\mathbf{Q}$ in powers of the spectral parameter k as $k\to\infty$. Hence, under certain additional conditions the system of equations (5) may prove to be either totally or partially integrable. The conditions can be formulated as restrictions imposed on the properties of the initial equation interpreted as an infinite-dimensional Hamiltonian system. Direct verification of these conditions is usually extremely tedious, however. There is another, simpler and more constructive, way of establishing these conditions, at least those involving the partial integrability of (1), i.e., the conditions in which Eq. (1) allows for multisoliton solution. It amounts to building for this equation a Lax representation, which is the starting point in applying the inverse scattering method to this equation. Let us see how this approach can be realized.

If we introduce into $r_k(x,t,u)$ an explicit dependence on the unknown function $u(x,t)$ and its derivatives, the first equation in (5) becomes the initial equation (1), and the second equation in (5) and Eq. (6) become the starting point for building the pair of operators of the Lax representation for (1).

Equation (6) is equivalent to the equations

$$\frac{\partial}{\partial x}\psi(x,t)=-Q_0(x,t),\quad \frac{\partial}{\partial t}\psi(x,t)=Q_1(x,t), \tag{7}$$

where $\psi(x,t)$ is a auxiliary function, which in a certain sense can be thought of as being the Wahlquist–Estabrook pseudopotential.[10,12] Indeed, since by introducing the dependence of ϕ on the spectral parameter k mentioned above we made the function ψ depend on k, the coefficients in the series expansion of this function in powers of k provide a denumerable set of pseudopotentials.

We introduce the auxiliary vector function

$$\mathbf{\Psi}(x,t)=\begin{pmatrix}\psi\\ \phi\end{pmatrix}.$$

The combination of the second equation in (5) and Eq. (6) can be written in the form of a system of two vector equations in the vector function $\mathbf{\Psi}(x,t)$,

$$\frac{\partial}{\partial x}\mathbf{\Psi}(x,t)=\mathbf{U}(x,t)\mathbf{\Psi}(x,t),$$

$$\frac{\partial}{\partial t}\ \mathbf{\Psi}(x,t)\mathbf{V}(x,t)\mathbf{\Psi}(x,t), \tag{8}$$

where $\mathbf{U}(x,t)$ and $\mathbf{V}(x,t)$ are 2-by-2 matrices. To this end we must augment the second equation in (5) and Eq. (6) by a relationship of the type

$$\frac{\partial}{\partial x}\phi(x,t)=a(x,t)\psi(x,t)+b(x,t)\phi(x,t), \tag{9}$$

where $a(x,t)$ and $b(x,t)$ are functions that have yet to determined. The introduction of such a large number of auxiliary functions is justified by the fact that the initial equation (1) and two additional equations for the auxiliary functions a and b comprise the condition for the compatibility of the pair of equations in (9).

The matrices $\mathbf{U}(x,t)$ and $\mathbf{V}(x,t)$ can easily be calculated:

$$\mathbf{U}(x,t)=\begin{pmatrix}0 & \varepsilon u\\ a & b\end{pmatrix},$$

$$\mathbf{V}(x,t)=\begin{pmatrix}r_2ua & u(b+r_{2x}-r_1)-u_xr_2\\ A/\varepsilon & B/\varepsilon\end{pmatrix}, \tag{10}$$

where

$$A(x,t)=a(br_2-r_1+2r_2)+a_xr_2,$$

$$B(x,t)=\varepsilon aur_2+b_xr_2+b(br_2-r_1+2r_{2x})$$
$$+r_{2xx}-r_{1x}+r_0.$$

The condition for compatibility of the pair of equations in (8) can be written in the form of the Zakharov–Shabat zero-curvature condition[1]

$$\frac{\partial}{\partial t}\mathbf{U}(x,t)-\frac{\partial}{\partial x}\mathbf{V}(x,t)+[\mathbf{U}(x,t),\mathbf{V}(x,t)]=0, \tag{11}$$

where the square brackets stand for the ordinary matrix commutator. Plugging $\mathbf{U}(x,t)$ and $\mathbf{V}(x,t)$ of (10) into (11) and doing a direct check, we conclude that Eqs. (11) are equivalent to (1) and the two additional equations

$$\varepsilon a_t-r_2a_{xx}+(r_1-3r_{2x})a_x+a(2r_{1x}-2b_xr_2-r_0-3r_{2xx}$$
$$-br_{2x})=0,$$

$$\varepsilon b_t-\frac{\partial}{\partial x}(b_xr_2+r_2b^2-(r_1-2r_{2x})b+2\varepsilon aur_2+r_0-r_x$$
$$+r_{2xx})=0. \tag{12}$$

Thus, we have found that an arbitrary equation of the form (1) can be represented in the form of a representation of the Lax type. However, this representation cannot be used directly in the inverse scattering method for building soliton solutions. This requires the spectral parameter to be included in the representation operators explicitly rather than formally. The general meaning of this requirement is that if such a parameter exists, by expanding the components of the vector **Q** in power series in the parameter and using (6), we can obtain an explicitly infinite sequence of conservation laws, which, as noted earlier, is necessary for complete integrability. The need for such a parameter can be justified more rigorously by using, for instance, the Gel'fand–Dikiĭ approach.[23,24]

According to the Gel'fand–Dikiĭ theory, the Hamiltonian nature and the integrability of equations with a Lax representation, which means the possibility of applying the inverse scattering method to such equations, are linked to the existence of a special expansion of the resolvent of one of the representation's operators in powers of the spectral parameter. Hence the first necessary condition for using the inverse scattering method to solve Eqs. (1) is an explicit dependence of the matrices $\mathbf{U}(x,t)$ and $\mathbf{V}(x,t)$ in the representation (8) on a complex-valued parameter λ that transforms the systems of linear equations (8) into a nontrivial system of spectral problems. Here the unknown function $u(x,t)$ is assumed independent of λ. This spectral parameter provides the expansions of the functions ψ and ϕ in the conservation law (5) that are needed for integrability, at least partial integrability.

3. CONSTRUCTING EQUATIONS THAT ALLOW FOR SOLITON SOLUTIONS

The only way to introduce into the system of equations (8) a spectral parameter (with $u(x,t)$ independent of λ) is to assume the existence of an explicit dependence on λ in the functions a and b: i.e., $a=a(x,t,\lambda)$ and $b=b(x,t,\lambda)$.

Gel'fand and Dikiĭ[23,24] showed that in order to relate a given first-order matrix operator to a new operator or a set of operators commuting with it and, as a consequence, a set of nontrivial nonlinear equations with a Lax representation, its matrix **U** must have the form

$$\mathbf{U}=\lambda \mathbf{D}(t)+\mathbf{U}_0(x,t) \tag{13}$$

with two conditions being imposed: (1) $\mathbf{D}=\mathrm{diag}(d_1(t),d_2(t))$ is a diagonal matrix with $d_1(t)\neq d_2(t)$, and (2) the diagonal elements of the matrix $\mathbf{U}_0$ vanish. In Refs. 23 and 24 it was assumed that $d_1\neq d_2=\mathrm{const}$. The main results in these papers do not change if d_1 and d_2 are dependent on t.

For the sake of convenience we say that a representation of the form (8) equipped with a spectral parameter is a true Lax representation or simply a Lax representation if at least one matrix of the representation has the form (13). Representations that do not obey this condition will be called pseudorepresentations or representations of the Lax type.

We start with the case where $r_2\equiv 1$, which corresponds to a situation in which the quadratic term in the dispersion of the medium is neither inhomogeneous nor nonlinear. Following (13), we put

$$a(x,t,\lambda)=\lambda a_1(x,t)+a_0(x,t),$$
$$b(x,t,\lambda)=\lambda b_1(x,t)+b_0(x,t). \tag{14}$$

Substitution of (14) into (10) and the transformation $\Psi\rightarrow\exp\{\lambda\theta(x,t)\}\Psi$ allow the matrix **U** in the form

$$\mathbf{U}=\lambda\mathbf{U}_1+\mathbf{U}_0,$$

where

$$\mathbf{U}_1(x,t)=\begin{pmatrix}\theta_x & 0\\ a_1 & b_1+\theta_x\end{pmatrix}. \tag{15}$$

The matrix $\mathbf{U}_1$ is a lower-triangle regular matrix and therefore can be reduced to diagonal form by a similarity transformation independent of λ, which induces a gauge transformation of the operators $\mathbf{L}_1=\partial_x-\mathbf{U}$ and $\mathbf{L}_2=\partial_t-\mathbf{V}$ of the Lax representation (Eqs. 8–10). In the process the matrices **U** and **V** are transformed as follows:

$$\mathbf{U}\rightarrow\mathbf{g}^{-1}\mathbf{U}\mathbf{g}-\mathbf{g}^{-1}\partial_x\mathbf{g},\quad \mathbf{V}\rightarrow\mathbf{g}^{-1}\mathbf{V}\mathbf{g}-\mathbf{g}^{-1}\partial_t\mathbf{g},$$

where

$$\mathbf{g}(x,t)=\begin{pmatrix}1 & 0\\ -a_1/(b_1+\theta_x) & 1\end{pmatrix}. \tag{16}$$

As a result $\mathbf{U}_1$ becomes a diagonal matrix, $\mathbf{U}_1=\mathrm{diag}\{d_1=\theta_x,d_2=b_1(x,t)+\theta_x\}$, and the matrix $\mathbf{U}_0$ becomes

$$\mathbf{U}_0(x,t)=\begin{pmatrix}-\varepsilon u a_1/b_1 & \varepsilon u\\ A_0 & B_0\end{pmatrix}, \tag{17}$$

where

$$A_0=a_0+\frac{\partial}{\partial x}\frac{a_1}{b_1}-\frac{a_1}{b_1}\left(\frac{a_1}{b_1}u+b_0\right),\quad B_0=\varepsilon\frac{ua_1}{b_1}+b_0.$$

Hence, if we assume that $\theta_x=\sigma=\mathrm{const}$ and Eq. (12) still allows for the solution $b_1=b_1(t)$, the gauge transformation with matrix (16) reduces the representation (8)–(10) to a form for which the first condition in (13) is met.

We perform the substitutions

$$u(x,t)=\tilde{u}(x,t)e^{\nu},\quad a_1(x,t)=\tilde{a}_1(x,t)e^{-\nu},$$
$$a_0(x,t)=\tilde{a}_0(x,t)e^{-\nu} \tag{18}$$

and at the same time assume that

$$\psi=\tilde{\psi}\exp\{-\xi(x,t)\},\quad \phi=\tilde{\phi}\exp\{-\xi(x,t)-\nu\}.$$

Here

$$\xi(x,t)=\int^x\frac{ua_1}{b_1}dx,\quad \nu(x,t)=-\int\left(\frac{2a_1u}{b_1}+b_0\right)dx.$$

As a result the matrix $\mathbf{U}_0$ has a zero diagonal and the matrix **U** assumes the form

$$\mathbf{U}=\lambda\begin{pmatrix}\sigma & 0\\ 0 & \sigma+b_1(t)\end{pmatrix}+\begin{pmatrix}0 & \varepsilon\tilde{u}\\ \tilde{A}_0 & 0\end{pmatrix}, \tag{19}$$

where $\tilde{A}_0=A_0(x,t)e^{\nu}$. This implies that for this operator the conditions for the applicability of the Gel'fand–Dikiĭ method are met. Gauge transformations do not alter the integrability of the equations. Hence the representation of the Lax type (8)–(10) corresponds, according to Ref. 24, to the partial-derivative representation of the equation of the general type (1), which makes it possible to use the inverse scattering method.

A prominent factor in the above constructions is that now we have a graphic way of relating the Lax representation to the structure of the initial equation. The function space that is conjugate to the solutions of the initial equation plays an important role in the structures of these representations. The representation operators act in the space of two-component functions Ψ, with one component being the function ϕ (the solution of the equation conjugate to the initial one) and the other the function ψ (the Wahlquist–Estabrook pseudopotential corresponding to the Lagrange identity). The existence of soliton solutions is observed if the nonlinearity of the original and adjoint equations is such that after conjugation the nonlinear terms of the initial equation are transformed into the nonlinear terms of the adjoint equation with, possibly, an additional explicit transformation of coordinates and the unknown functions.

4. ONE-WAVE EQUATIONS IN MEDIA WITH QUADRATIC NONLINEARITY

The equations corresponding to the conditions (14) can be derived directly by plugging (14) into (12). If by analogy to (18) we put

$$u(x,t)=\tilde{u}(x,t)e^{\mu}, \quad a_1(x,t)=\tilde{a}_1(x,t)e^{-\mu},$$
$$a_0(x,t)=\tilde{a}_0(x,t)e^{-\mu}, \tag{20}$$

where

$$\mu(x,t)=-\int b_0(x,t)\, dx,$$

we arrive at the following equations for the functions $\tilde{u}(x,t)$, $\tilde{a}_0(x,t)$, and $\tilde{a}_1(x,t)$ (we drop the tildes for brevity):

$$\varepsilon u_t+u_{xx}+[r_1(t)-\varepsilon q(t)x]u_x+2\varepsilon p(t)\frac{\partial}{\partial x}(a_1u^2)$$
$$-2\varepsilon a_0u^2+[r_2(t)-\varepsilon q_1(t)]u=0,$$
$$-\varepsilon a_{0t}+a_{0,xx}-[r_1(t)-\varepsilon q(t)x]a_{0,x}-2\varepsilon p(t)$$
$$\times\frac{\partial}{\partial x}(a_0a_1u)-2\varepsilon a_0^2u+[r_2(t)+\varepsilon q(t)]a_0=0,$$
$$-\varepsilon a_{1,t}+a_{1,xx}-(r_1(t)-\varepsilon q(t)x)a_{1,x}-2\varepsilon p(t)$$
$$\times\frac{\partial}{\partial x}(a_1^2u)-2\varepsilon a_0a_1u+r_2(t)a_1=0. \tag{21}$$

Here $r_1(t)$, $r_2(t)$, and $p(t)$ are arbitrary functions of t, and

$$q(t)=\frac{d}{dt}\ln p(t).$$

The meaning of the substitution (20) is that the function $b_0(x,t)$ is eliminated from the equations, which can be considered a deformation of Eqs. (21) to standard form.

Generally, the analysis of the gauge transformations that reduce the initial Lax representation to the standard one can be considered a deformation in the sense of Ref. 17. Here the function $b(x,t,\lambda)=b_1(t)\lambda+b_0(x,t;u)$ is simply a spectral parameter of general form, which depends on position and time and, possibly, on the unknown function $u(x,t)$. From this viewpoint the suggested method provides almost all possible deformations of the standard representation. This can easily be verified if we seek the solution of Eqs. (12) with respect to the functions $a(x,t,\lambda)$ and $b(x,t,\lambda)$ in the form of polynomials of an arbitrary finite degree in λ whose coefficients depend on x and t. In this case all solutions are reduced to those of Eqs. (21). Other solutions are possible only in the limit where the degree of the polynomial goes to infinity. Equations of this type, if they exist, apparently comprise a special class of equations.

After the additional reduction

$$a_0(x,t)=i\alpha u^*(x,t), \quad a_1(x,t)=i\beta u^*(x,t),$$
$$\varepsilon=i,\alpha,\beta=\text{const}, \quad q(t)=iQ(t), \quad r_1(t)=iR_1(t),$$
$$r_2(t)=R_2(t), \quad P(t)=\exp\left\{i\int Q(t)\, dt\right\} \tag{22}$$

the equations for the complex-valued function $u(x,t)$ assume the form

$$iu_t+u_{xx}+i[R_1(t)-Q(t)x]u_x-2i\beta P(t)\frac{\partial}{\partial x}(|u|^2u)$$
$$-2\alpha|u|^2u+[R_2(t)-iQ(t)]u=0 \tag{23}$$

with variable coefficients for an arbitrary dependence on t of the functions $R_1(t)$, $R_2(t)$, and $Q(t)$. This equation is encountered in nonlinear optics in problems of Raman scattering (see, e.g., Refs. 21,25, and 26) in media with quadratic dispersion and cubic nonlinearity.

One of the typical examples that generalize the description of wave propagation in such media when the refractive index of the medium varies in space and time is the nonlinear Schrödinger equation with a parabolic profile of the refractive index.[27] This type of equations can be obtained from (23) via the reduction $\beta=0$ and an additional substitution $p=u\exp\{-iQ(t)x\}$.

The general form of Eqs. (23) can be broadened still further by the following observation. Since the function $\nu(x,t)$ in Eqs. (15) is arbitrary, there is a class of equations related to Eq. (22) by the transformations (18) (deformations) in which $\nu=\nu(x,t,|u|)$. If the equations for u, a_0, and a_1 together yield a certain conservation law, the law can be identified with the equation for b_0. Let us put $Q(t)\equiv 0$. Then the simplest conservation law identified with the equation for b_0 yields

$$\nu(x,t)=i\sigma\int |u|^2\, dx,$$

which leads to the Eckhause equation (see, e.g., Ref. 28)

$$iu_t+u_{xx}+iR_1(t)u_x+\sigma|u|^4u-2\alpha|u|^2u$$
$$+2i[(\sigma-\beta)\,\partial_x(|u|^2u)-\sigma|u|^2u_x]=0. \tag{24}$$

At the same time the function

$$p=u\ \exp\left\{-i\sigma\int |u|^2\ dx\right\}$$

satisfies Eq. (23) at $Q(t)\equiv 0$. Obviously, a similar procedure is possible for any other conservation law for Eq. (23), so that there is an hierarchy of equations linked to Eq. (23) by relationships of the form

$$p=u\ \exp\left\{-i\int I(u,\bar{u})\ dx\right\},$$

where $I(u,\bar{u})$ is a conserved density of the equation for u.

5. NONLINEARITY AND INHOMOGENEITY OF QUADRATIC DISPERSION

In the case $r_2(x,t)\not\equiv 1$, i.e., when the medium has such properties that the value of the quadratic term in the dispersion of the medium may depend on position and time (inhomogeneity; see, e.g., Refs. 2 and 3) and on the parameters of the wave process proper (nonlinearity), it is possible to reduce the problem of describing all models that allow for soliton solutions in media of this type to the previous case.

To this end we perform the following change of variables in the functions in (1):

$$u(x,t)=v(\theta(x,t),t),$$

$$r_k(x,t;u)=q_k(\theta(x,t),t;v), \quad k=1,2,\ldots.$$

In terms of the new variables, when

$$r_2(x,t;u)=\theta_x^{-2}, \tag{25}$$

the initial equation becomes

$$\varepsilon v_t+v_{\theta\theta}+R_1(\theta,t;v)v_\theta+R_0(\theta,t;v)v=0, \tag{26}$$

where

$$R_0(\theta,t;v)=q_0(\theta,t;v),$$

$$R_1(\theta,t;v)=q_2\theta_{xx}-\varepsilon\theta_t+q_1\theta_x, \quad R_2\equiv 1.$$

Since the existence of multisoliton solutions does depend on the choice of the independent variables, by applying the above procedure to Eq. (26) we can obtain all the types of nonlinear equations with an inhomogeneous quadratic dispersion law (the linear part) that allow for soliton solutions. Here Eq. (25) determines the characteristics along which the solitons move in the x,t plane. If the coefficient $r_2(x,t;u)$ explicitly depends on $u(x,t)$, this equation and the equation $u(x,t)=v(\theta,t)$ must be solved simultaneously. In this case the characteristic depends on the solution.

Note that in contrast to Refs. 2 and 3, the given method provides the conditions for the existence in a medium with variable parameters not of a single wave of the soliton type but of an arbitrary set of waves interacting elastically, which actually is the definition of the very concept of a soliton. In comparison to the solitons of "unperturbed" equations, the solitons of the above equations can also change their parameters, such as the amplitude and width. However, this evolution is in strict compliance with the requirement that the solitons interact elastically with each other.

6. EQUATIONS FOR MEDIA WITH HIGHER-ORDER DISPERSION

There is little that has to be modified in the given method so it can be used to study the types of nonlinearity of a one-dimensional medium in which solitary waves can propagate and in which the dispersion order is higher than two, e.g., in the case of cubic dispersion. If the order of dispersion is N, the general equation describing the variation of the complex-valued amplitude of a weakly nonlinear wave in such a medium can be written as

$$\varepsilon u_t+\sum_{k=0}^{N} r_k(x,t;u)u^{[k]}=0, \tag{27}$$

where

$$u^{[k]}=\frac{\partial^k}{\partial x^k}u(x,t).$$

Among the equations that belong to this type are, for instance, the Korteweg–de Vries equations and the modified Korteweg–de Vries equations. The suggested scheme for such equations does not involve serious modifications—only the volume of calculations increases. As an example that is important for applications we give the general form of the matrices of the Lax pseudorepresentation for the case $N=3$. Here the pair of operators of the Lax pseudorepresentation, constructed in a way similar to the above case, has the same general form (8), the only difference being that the elements of the matrix $\mathbf{V}$ have a more complicated dependence on the coefficients $r_k(x,t)$ of $\mathbf{L}$. The matrix $\mathbf{U}$ has the same form as in (10). For $N=3$ and $r_3(x,t)\equiv 1$,

$$\mathbf{U}(x,t)=\begin{pmatrix} 0 & \varepsilon u \\ a & b \end{pmatrix}, \qquad \mathbf{V}(x,t)=\begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & D(x,t) \end{pmatrix}, \tag{28}$$

where

$$A(x,t)=au(-b+r_2)+au_x-a_xu,$$

$$B(x,t)=-\varepsilon au^2+(-b_x+r_{2x})u+(bu-u_x)(-b+r_2)$$
$$-u_{xx}+r_1u,$$

$$C(x,t)=-\varepsilon a^2u+2(-b_x+r_{2x})a+(ba-a_x)(-b+r_2)$$
$$-a_{xx}-r_1a,$$

$$D(x,t)=-\varepsilon a(2bu+u_x-r_2u)-b^2(b-r_2+3b_x-2r_2$$
$$-r_1)-2\varepsilon a_xu-b_{xx}+b_xr_2-r_{1x}+r_{2xx}+r_0. \tag{29}$$

After we substitute (14) in the Zakharov–Shabat equations (11) and perform transformations similar to (20), the equations become

$$\varepsilon u_t+u_{xxx}+r_1(t)u_{xx}+[r_2(t)-\varepsilon q(t)x]u_x$$
$$+2\varepsilon r_1(t)\frac{\partial}{\partial x}(a_1u^2)-2\varepsilon r_1(t)a_0u^2+6\varepsilon\frac{\partial}{\partial x}(a_1uu_x)$$
$$-3\varepsilon a_0\frac{\partial}{\partial x}u^2+6\varepsilon^2\frac{\partial}{\partial x}(a_1^2u^3)-6\varepsilon^2a_0a_1u^3$$

$$+[r_3(t)-\varepsilon q(t)]u=0,$$

$$\varepsilon a_{0t}+a_{0xxx}-r_1(t)a_{0xx}+[r_2(t)-\varepsilon q(t)x]a_{0x}$$

$$+2\varepsilon r_1(t)\frac{\partial}{\partial x}(a_0a_1u)+2\varepsilon r_1(t)a_0^2u$$

$$-3\varepsilon\frac{\partial}{\partial x}(u(a_0a_1)_x)-3\varepsilon u\frac{\partial}{\partial x}a_0^2+6\varepsilon^2\frac{\partial}{\partial x}(a_0a_1^2u^2)$$

$$-6\varepsilon^2a_0^2a_1u^2-[r_3(t)+\varepsilon q(t)]a_0=0,$$

$$\varepsilon a_{1t}+a_{1xxx}-r_1(t)a_{1xx}+[r_2(t)-\varepsilon q(t)x]a_{0x}$$

$$+2\varepsilon r_1(t)\frac{\partial}{\partial x}(a_1^2u)+2\varepsilon r_1(t)a_0a_1u$$

$$-6\varepsilon\frac{\partial}{\partial x}(ua_1a_{1x})-3\varepsilon u\frac{\partial}{\partial x}(a_0a_1)+6\varepsilon^2\frac{\partial}{\partial x}(a_1^3u^2)$$

$$+6\varepsilon^2a_0a_1^2u^2-r_3(t)a_1=0. \qquad (30)$$

Here $r_1(t)$, $r_2(t)$, $r_3(t)$, and $q(t)$ are arbitrary functions of t. By appropriate reduction these equations are transformed into the Korteweg–de Vries equation, the modified Korteweg–de Vries equation, and a set of equations encountered in nonlinear optics when the cubic dispersion of the medium is taken into account.[21]

All the main conclusions of Secs. 3 and 4 concerning equations with quadratic dispersion remain valid when applied to equations of the form (27). For instance, the gauge transformations that transform the matrix **U** into (13) have the same form as in (15) and (16). When $r_n(x,t)\neq 1$ holds, the conclusions of Sec. 4 remain valid, the only difference being that now the equation for the characteristics assumes the form

$$r_n(x,t;u)=\theta_x^{-n}.$$

The coefficients of the equation with the independent variable $\theta=\theta(x,t)$ can be calculated directly by replacing x with $\theta(x,t)$. For instance, for $N=3$

$$R_0=r_0,\quad R_1=r_3\theta_{xxx}+r_2\theta_{xx}+r_1\theta_x-\varepsilon\theta_t,$$

$$R_2=3r_3\theta_x\theta_{xx}+r_2\theta_x^2,\quad R_3=1.$$

An example of an equation of this type is the well-known equation $u_t=u^3u_{xxx}$.

7. EQUATIONS WITH D'ALEMBERT AND LAPLACE OPERATORS

To a great extent the above examples serve as a demonstration of the calculation techniques used in the proposed method, since most of the equations obtained here are known. This is especially true of the equations with a quadratic dispersion law, for which Refs. 11,13 and 15 list the completely integrable scalar equations. Less studied and more diversified (and hence more complicated for investigation) is the case where the equations contain a d'Alembertian or Laplacian operator as the linear dispersion part. Such problems are often encountered in practice. Among these are, for instance, the sine-Gordon equation and its modifications. Here we touch on the application of the suggested approach to equations of this type, which may serve as a useful example for comparison with the method of deformation of the Lax representations.[17]

Let us examine an equation of the type

$$u_{xt}+q(x,t;u)u_t+r(x,t;u)u_x+s(x,t;u)u=0.$$

The suggested scheme for constructing a Lax representation for this class of equations leads to a pair of operators of the following form:

$$\mathbf{U}(x,t)=\begin{pmatrix} au & v(x,t)\\ a & b\end{pmatrix},$$

$$\mathbf{V}(x,t)=\frac{1}{\Lambda(x,t)}\begin{pmatrix} A(x,t) & B(x,t)\\ C(x,t) & D(x,t)\end{pmatrix}, \qquad (31)$$

where

$$A(x,t)=u(a_t-ar)/2,\quad B(x,t)=(b-g)g-uf/2,$$

$$C(x,t)=ar-a_t,\quad D(x,t)=f-ag,$$

$$\Lambda(x,t)=b-au/2-q,\quad v(x,t)=(b-2q)u/2-u_x/2,$$

$$f(x,t)=br+r_x+q_t-b_t-s,\quad g=u_t/2+ru. \qquad (32)$$

The most drastic difference from the previous examples is that the auxiliary functions $a(x,t,\lambda)$ and $b(x,t,\lambda)$ are in the denominator of **V**. This reveals the explicit analogy with the representation with a variable spectral parameter,[17] where the function Λ is the parameter. The equations for $a(x,t,\lambda)$ and $\Lambda(x,t,\lambda)$ (where $b=\Lambda+q+\frac{1}{2}au$), which follow from the Zakharov–Shabat equations, have the form

$$-a_{xt}-a(ua)_t+2ar_x-sa+qa_t+ra_x-a\Lambda_t+(a_t-ar)\Lambda_x/\Lambda=0,$$

$$\Lambda_{xt}\Lambda-\Lambda_x\Lambda_t+\Lambda_t\Lambda^2-\Lambda_x(au_t-aur-r_x+qr-s)$$

$$+\Lambda[aur_x-au_tq-asu+a_xu_t+rua_x-(qr)_x-r_{xx}$$

$$+s_x]+\Lambda^2[(au)_t+q_t-r_x]=0. \qquad (33)$$

Setting $\Lambda(x,t)=$const, we arrive at a system of equations whose Lax representation contains the spectral parameter λ and, which can be reduced to the "standard" Gel'fand–Dikiĭ form (19) by gauge transformations. The system is

$$u_{xt}+qu_t+ru_x+su=0,\quad a_{xt}-(qa)_t-(ra)_x+sa=0,$$

$$(au)_t+q_t=r_x,\quad (ar)_xu-aqu_t-aus$$

$$+a_xu_t+\frac{\partial}{\partial x}(s-qr-r_x)=0 \qquad (34)$$

and contains one arbitrary functional parameter, one of the functions $q(x,t;u)$, $r(x,t;u)$, or $s(x,t,u)$. This arbitrariness is similar to that in the choice of the function b_0 in the examples considered earlier and can be eliminated by the gauge transformations examined in Sec. 3. Indeed, the matrix **U** in this case can be written as

$$\mathbf{U}=\lambda\mathbf{U}_1+\mathbf{U}_0,$$

where

$$\mathbf{U}_1(x,t)=\begin{pmatrix} 0 & u/2 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{U}_0(x,t)=\begin{pmatrix} au/2 & au^2/4-u_x/2-qu/2 \\ a & au/2+q \end{pmatrix}.$$

The matrices $\mathbf{U}_1$ and $\mathbf{U}_0$ and the matrices of the gauge transformations become the corresponding matrices of (15)–(19) if we perform the substitutions

$$u\rightarrow a_0, \quad au^2/4-u_x/2-qu/2\rightarrow u, \quad u/2\rightarrow a_1, \quad q\rightarrow b_0$$

and the additional transformation

$$\Psi\rightarrow\Psi\exp\left\{\int\frac{au}{2}\,dx\right\}.$$

Note that in the case of complex-valued variables Eq. (19) allows for an interesting reduction: $a=i\kappa u^*$. In this case the equation for a is the complex-conjugate of the equation for u and the functions q, r, and s have the following form:

$$s=g(x,t)+i\sigma_t(x,t), \quad r=\frac{\partial}{\partial t}\int\left(\sigma+\frac{1}{2}\,\kappa|u|^2\right)\,dx+h(t),$$

$$q=\sigma-\tfrac{1}{2}\kappa|u|^2+f(x),$$

where g and σ are new unknown real-valued functions, and $f(x)$ and $h(t)$ are arbitrary real-valued functions. The function σ can be eliminated from the equations by the substitution

$$u=v(x,t)\exp\left(-i\int\sigma\,dx\right),$$

after which we are left with an equation for v of the form

$$v_{xt}+iv_t\left[f(x)-\frac{1}{2}\,\kappa|v|^2\right]+iv_x\,\frac{\partial}{\partial t}\times\int\left(\frac{1}{2}\,\kappa|v|^2\right)\,dx+gv=0. \tag{35}$$

The equation for g can easily be obtained and integrated. It contains nonlinearities of the second and fourth orders in u. However, due its cumbersomeness we do not write it here. An interesting aspect of Eq. (35) is that it describes the propagation of electromagnetic waves (in conical variables) in a medium with a cubic nonlinearity of a certain type without the parabolic-equation approximation. The study of this equation constitutes a separate interesting problem.

These transformations and other possible function representations of Λ (for instance, $\Lambda=\lambda+L_0(x,t)$, can be considered deformations of Eqs. (34), whose set is apparently more rich than the set of deformations of the evolutionary equations considered above. We see that, in contrast to Ref. 17 and other papers, this set of deformed equations follows immediately from Eqs. (33) after we have explicitly fixed the function form of the spectral-parameter variable and requires no further calculations. For all deformations of this type the Lax representation is defined simultaneously and can be used in the inverse scattering problem.

8. INTERACTION OF WAVES IN DISPERSIVE MEDIA

Another rich and, therefore, poorly studied class of equations is that of multicomponent equations allowing for soliton solutions. A complete list of such equations has yet to be put together, but they are important from the practical angle. Let us see how the proposed scheme is modified in the case of multicomponent nonlinear equations, for instance, the equations describing three-wave interactions[18] and, in general, the interaction of N waves. The statement of the problem for this case is as follows. Suppose that initially M almost-periodic waves propagate in a medium. The waves have different, but fixed, wave numbers and frequencies, and their amplitudes slowly vary in space and time: $a_i=a_i(x,t)$, $i=1,2,\ldots,M$. If the medium is nonlinear, these primary waves interact with each other and generate a spectrum of new waves with other frequencies and wave numbers. In many cases that are important in practice this spectrum contains only harmonic waves whose parameters obey strict conditions of synchronism with the parameters of the primary waves and with each other. In view of this the number K of the generated waves proves to be finite, and the amplitudes a_j, $j=M+1,M+2,\ldots,M+K$, of these waves are coupled by a finite number of equations and depend on the amplitudes of the primary waves. The representation of the total perturbation field in this case is

$$\mathscr{E}(x)=\sum_{m=1}^{M+K}\left[a_m(x,t)\exp\{i(k_mx-\omega_mt)\}+a_m^*(x,t)\exp\{-i(k_mx-\omega_mt)\}\right].$$

Generally, the equations of the slow evolution of the amplitudes of the primary waves can be written as

$$\mathbf{L}_ia_i=\sum_{j=1}^{M}w_{ij}a_j, \tag{36}$$

where the operators $\mathbf{L}_i$ describe the dispersive properties of the medium, and the matrix $\mathbf{W}=(w_{i,j})$ the interaction of the primary waves with each other and with the generated secondary waves. Basically the operators $\mathbf{L}_i$ and the matrices $\mathbf{W}$ are determined by the properties of the medium in which the wave process takes place and the geometry of the boundary of the medium if the waves have the meaning of excitations of various modes in cavities or waveguides. Actually the separation of the waves into primary and secondary is purely arbitrary and can be done only by specifying the initial and boundary conditions for the various almost-periodic components of the wave process.

For such systems we can also pose the problem of finding all types of media for which the system of equations (36) allows for multisoliton excitations, which usually are stable localized excitations.

We seek the solution to this problem by employing the Lagrange identity by analogy with Ref. 18, using the example of N interacting waves propagating, however, in a medium with a quadratic dispersion law. In this case the evolution of a system of n primary waves can be written as

$$\partial_t a_i + v_i \partial_x a_i + d_i \partial_x^2 a_i = \sum_{j=1}^{n} w_{ij} a_j, \quad i,j = 1, \ldots, n. \tag{37}$$

Here the $v_i(x,t)$ are the group velocities of the primary waves, and the $d_i(x,t)$ are the coefficients of the quadratic term in the law of dispersion for frequency. In addition to (37) we examine the adjoint system of equations

$$-\frac{\partial \phi_i}{\partial t} - \frac{\partial(v_i(x,t), \phi_i)}{\partial x} + \frac{\partial^2 (d_i \phi_i)}{\partial x^2} = \sum_{j=1}^{n} w_{ji} \phi_j, \quad i,j = 1, \ldots, n. \tag{38}$$

Here the ϕ_i are the adjoints of the functions a_i. Forming the left scalar product of Eqs. (37) and ϕ_i and the right scalar product of Eqs. (38) and a_i and subtracting the products, we arrive at the following generalized conservation law

$$\frac{\partial}{\partial t} \sum_{i=1}^{n} \phi_i a_i + \frac{\partial}{\partial x} \sum_{i=1}^{n} [v_i(x,t) \phi_i a_i + d_i \phi_i \partial_x a_i - a_i \partial_x (d_i \phi_i)] = 0, \tag{39}$$

which is automatically satisfied if

$$\frac{\partial \psi}{\partial x} = \sum_{i=1}^{n} \phi_i a_i,$$

$$\frac{\partial \psi}{\partial t} = -\sum_{i=1}^{n} [v_i(x,t) \phi_i a_i + d_i \phi_i \partial_x a_i - a_i \partial_x (d_i \phi_i)] \tag{40}$$

for an arbitrary differentiable function $\psi(x,t)$. As before, the emergence of a conservation law as a result of combining adjoint equations is a consequence of the Lagrange identity.

Let us now take an auxiliary vector function $\mathbf{\Psi} = (\psi, \phi_1, \phi_2, \ldots, \phi_n)^T$, auxiliary n-by-n matrices with the elements b_{ij}, and vectors c_i such that

$$\frac{\partial \phi_i}{\partial x} = \sum_{i=1}^{n} b_{ij} \phi_j + c_i \psi, \quad i,j = 1, \ldots, n. \tag{41}$$

Then the combination of Eqs. (38), (39) and (41) can be written in the form of a system of two vector equations in the vector function $\mathbf{\Psi}(x,t)$,

$$\frac{\partial}{\partial x} \mathbf{\Psi} = \mathbf{U} \mathbf{\Psi}, \quad \frac{\partial}{\partial t} \mathbf{\Psi} = \mathbf{V} \mathbf{\Psi}, \tag{42}$$

where $\mathbf{U}$ and $\mathbf{V}$ are two $((n+1)$-by-$(n+1))$ matrices of the form

$$\mathbf{U}(x,t) = \begin{pmatrix} 0 & a_1 & a_2 & \cdots & a_n \\ c_1 & b_{11} & b_{12} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n & b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix},$$

$$\mathbf{V}(x,t) = \begin{pmatrix} D & A_1 & A_2 & \cdots & A_n \\ C_1 & B_{11} & B_{12} & \cdots & B_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_n & B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix}, \tag{43}$$

with

$$A_i = \sum_{j=1}^{n} (d_j a_j b_{ji}) - (v_i - d_{i,x}) a_i - d_i a_{i,x},$$

$$C_i = d_i \sum_{j=1}^{n} (c_j b_{ij}) - (v_i - 2d_{i,x}) c_i + d_i c_{i,x},$$

$$B_{ij} = -w_{ji} - (v_i - 2d_{i,x}) b_{ij} - (v_{i,x} - d_{i,xx}) \delta_{ij} + d_i b_{ij,x} + d_i c_i a_j + d_i \sum_{k=1}^{n} (b_{ik} b_{kj}),$$

$$D = \sum_{i=1}^{n} (d_i a_i c_i), \quad i,j = 1, \ldots, n.$$

The complete set of Zakharov–Shabat equations (11) corresponding to the matrices (43) is

$$a_{i,t} - A_{i,x} + \sum_{k=1}^{n} (a_k B_{ki} - A_k b_{ki}) - D a_i = 0,$$

$$c_{i,t} - C_{i,x} - \sum_{k=1}^{n} (c_k B_{ik} - C_k b_{ik}) + D c_i = 0,$$

$$b_{ij,t} - B_{ij,x} + \sum_{k=1}^{n} (b_{ik} B_{kj} - B_{ik} b_{kj}) + c_i A_j - C_i a_j = 0,$$

$$i,j = 1, \ldots, n. \tag{44}$$

The equations for a_i and c_i prove to be self-adjoint, so that the c_i can be interpreted as being the conjugate amplitudes of the primary waves. The equations for b_{ij} and b_{ji} with $i \neq j$ are also adjoints, with the result that the off-diagonal elements b_{ij} can be interpreted as being the amplitudes of the secondary waves.

To find the equations that allow for soliton solutions we must generally assume that

$$b_{ij} = \lambda b_{ij}^{(1)}(x,t) + b_{ij}^{(0)}(x,t),$$

$$c_i = \lambda c_i^{(1)}(x,t) + c_i^{(0)}(x,t), \quad i,j = 1, \ldots, n. \tag{45}$$

As in the case of one-wave equations, there is a gauge transformation that transforms the matrix $\mathbf{U}$ into (13), which makes it possible to construct the solutions to the emerging equations by applying the standard techniques of the inverse scattering problem.

For an example of wave interaction we take the model, often used in practical applications, of three-wave interaction in a medium with quadratic dispersion with $d_1(x,t) = \text{const}$ and $d_2(x,t) = \text{const}$. Here $n = 2$, and $c_i^{(1)} = 0$ and $b_{ij}^{(1)} = 0$ hold for $i \neq j$. As a result of plugging (45) into the equations (44) and allowing for all the restrictions we arrive at the following relationships for the medium parameters and the elements of $\mathbf{W}$:

$$d_1 = d_2 = d, \quad b_{11}^{(1)} = \sigma_1, \quad b_{22}^{(1)} = \sigma_2,$$

$$b_{11}^{(0)}(x,t) = \frac{v_1(x,t)}{2d} + q_1(t),$$

$$b_{22}^{(0)}(x,t) = \frac{v_2(x,t)}{2d} + q_2(t),$$

$$v_1(x,t) = 2P_{1,x}, \quad v_2(x,t) = 2P_{2,x}.$$

We introduce the notation $a_3=b_{12}$ and $c_3=b_{21}$. Then for $d=i$, after the reduction

$$a_1=i\tilde{a}_1\exp\{-iP_1\},\quad a_2=i\tilde{a}_2\exp\{-iP_2\},$$

$$a_3=i\tilde{a}_3\exp\{-i(P_2-P_1)\},\quad c_1=-i\tilde{a}_1^*\exp\{iP_1\},$$

$$c_2=-i\tilde{a}_2^*\exp\{iP_2\},\quad c_3=-i\tilde{a}_3^*\exp\{i(P_2-P_1)\}$$

the equations for three-wave interaction assume the form

$$a_{1,t}+ia_{1,xx}-2\gamma a_2a_{3,x}^*-2i(|a_1|^2+\gamma|a_3|^2)a_1 +i\dot{q}_1xa_1=0,$$

$$a_{2,t}+ia_{2,xx}-2(\gamma-1)a_1a_{3,x} -2i[|a_2|^2+(\gamma-1)|a_3|^2]a_2+i\dot{q}_2xa_2=0,$$

$$a_{3,t}+i(1-2\gamma)a_{3,xx}-2\frac{\partial}{\partial x}(a_2a_1^*) -2i[|a_1|^2-|a_2|^2+(2\gamma-1)|a_3|^2]a_3 +\delta a_{3,x}+i(\dot{q}_1-\dot{q}_2)xa_3=0,$$

where the tilde on the complex-valued amplitudes has been dropped and the following notation has been introduced:

$$\gamma=\frac{\sigma_1}{\sigma_1-\sigma_2},\quad \delta(t)=q_1(t)(\gamma-1)-q_2(t)\gamma.$$

Here $q_1(t)$ and $q_2(t)$ are arbitrary real-valued functions of t. Usually balancing nonlinearity and dispersion in media with a quadratic dispersion law requires a cubic nonlinearity. In addition to containing terms with a cubic nonlinearity the present equations have terms with a quadratic nonlinearity, which is a characteristic feature of the three-wave interaction equations in the case of linear dispersion. We also note that in a particular case the given system contains equations obtained earlier by Khasilev,[29] who developed a Lax representation for the equations of the two-frequency interaction of waves propagating in a medium with a quadratic dispersion law and a cubic nonlinearity.

A remarkable feature of this system of equations is that they contain an arbitrary parameter γ, and the structure of the equations describing the behavior of the secondary wave with the amplitude a_3 strongly depends on the value of this parameter. At $\gamma=\frac{1}{2}$ the dispersion term disappears from the equation for a_3 and the secondary wave propagates without dispersion. This situation can occur in realistic conditions[22,30] in an anomalous-dispersion region. Other "degenerate" variants of these equations, say, the equations describing second-harmonic generation with allowance for quadratic dispersion, are also of interest.

9. CONCLUSION

The above examples and additional investigations suggest that the proposed scheme for constructing models that allow for soliton solutions is fairly universal. All known one-dimensional soliton equations can be found within this scheme. In addition, the scheme makes it possible to construct various modifications of known equations by starting at specific physical conditions imposed on the model of the process by requirements of the dispersion order and the very existence of soliton solutions. The merit of the present method is that it begins with the specific form of the equation being investigated and, in the event of success, immediately yields the operators of the Lax representation in explicit form, which makes it possible to employ the inverse scattering method. Moreover, within this approach all possible modifications (deformations) of the equation allowed by the existence of soliton solutions are obtained simultaneously. This determines the practical value of such an approach.

To stress the universal nature of the proposed approach we point to the natural possibility of applying the ideas of the method to the multidimensional case. In the most general case the method makes it possible to examine models of wave interaction in n-dimensional media with a given order of dispersion in each coordinate and to find Lax pseudorepresentations for these models.

The formal calculation of the n matrices of the Lax pseudorepresentation for such model is easy. But finding the conditions in which such pseudorepresentations correspond to true representations is quite a different matter. In addition to purely technical difficulties (lengthy calculations), it is difficult to select a dependence of the auxiliary functions on the spectral parameter for which the inverse spectral problem can be successfully solved. The difficulty here lies in the fact that the form of the nonlinearity in the equations that allow for soliton solutions is determined not by the general functional shape of the dispersion hypersurface in the space of the wave numbers of its linear part but rather by the shape of each dispersion curve parametrized by a single spectral parameter, i.e., for each dispersion curve we must establish whether the respective soliton equation exists and find the type of nonlinearity in this equation. The equations obtained by the Lagrange identity method are not sufficiently selective in, and sensitive to, the shape of the dispersion curve and therefore carry information above several equations with different types of nonlinearity. Separating these equations constitutes the main problem, which requires other methods for its solution.

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