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## ПРИМЕНЕНИЕ УРАВНЕНИЯ ШРЁДИНГЕРА В КОСМОЛОГИИ СО СКАЛЯРНЫМ ПОЛЕМ НА ОСНОВЕ ТОЧНЫХ РЕШЕНИЙ*

Фомин И. В.[a,1], Червон С. В.[a,b,2], Махарадж С. Д.[c,3]

[a] Московский государственный технический университет им. Н.Э. Баумана, г. Москва, 105005, Россия.

[b] Ульяновский государственный педогогический университет, г. Ульяновск, 432071, Россия.

[c] Университет Квазулу–Натал, г. Дурбан, 4000, Южная Африка.

Предложен новый метод построения точных решений космологии скалярного поля, основанный на представлении динамических уравнений Эйнштейна–Фридмана в виде уравнения Шрёдингера. Это представление позволяет сравнивать решения квантово-механических и космологических задач. С другой стороны, этот подход позволяет использовать известные форм-инвариантные преобразования уравнения Шрёдингера для генерации точных космологических решений. В качестве примера применения данного метода рассмотрено использование преобразований Дарбу в космологии со скалярным полем. С другой стороны, представленные методы позволяют обобщить полученные решения на многополевые космологические модели.

*Ключевые слова*: уравнение Шрёдингера, инфляция, точные решения.

## APPLICATION OF THE SCHRÖDINGER EQUATION IN EXACT SCALAR FIELD COSMOLOGY

Fomin I. V.[a,1], Chervon S. V.[a,b,2], Maharaj S. D.[c,3]

[a] Bauman Moscow State Technical University, Moscow, 105005, Russia.

[b] Ulyanovsk State Pedagogical University, Ulyanovsk, 432071, Russia

[c] University of KwaZulu–Natal, Durban 4000, South Africa.

We propose a new method of exact solutions construction for scalar field cosmology based on representation of the Einstein-Friedmann dynamic equations as Schrödinger-like one. This representation allows one to compare the solutions of quantum-mechanical and cosmological problems. On the other hand, this approach makes it possible to use the well-known form-invariant transformations of the Schrödinger equation to generate exact cosmological solutions. As an example of the application of this method, the use of the Darboux transformations in scalar field cosmology is considered. On the other hand, the presented methods make it possible to generalize the obtained solutions to multi-field cosmological models.

*Keywords*: Schrödinger equation, inflation, exact solutions.

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### Introduction

Investigation of Scalar Field Cosmology (SFC) is closely connected with the development of inflationary theory started in the beginning of 1980ies with works by Starobinsky, Guth, Linde and

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[1]E-mail: ingvor@inbox.ru
[2]E-mail: chervon.sergey@gmail.com
[3]E-mail: maharaj@ukzn.ac.za

Albrecht and Steinhard [1–5]. The first analysis of the system of differential equations describing the dynamics of the Friedmann universe filled with a scalar field was performed using approximation methods. About ten years after the discovery of the inflationary stage in the evolution of the universe, the first exact solution attracted attention of many scientists. Since that time a great number of methods for construction of exact solutions in SFC have been proposed and developed. Many of these are described in the works [6–21].

The equations of cosmological dynamics themselves in inflationary models with a scalar field in the flat Friedmann universe are written as follows

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{1}$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2, \tag{2}$$

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi). \tag{3}$$

Here $a(t)$ is the scale factor, $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter, $\phi(t)$ is a scalar field, and $V(\phi)$ is a potential energy (or simply *potential* as it traditionally described in inflationary cosmology). A dot denotes the derivative with respect to the cosmic time $t$, and a prime denotes the derivative with respect to the scalar field.

It should also be noted that field equation (3) is a consequence of two Einstein-Friedman equations (1)–(2), which completely determine the dynamics of the early universe at the inflationary stage based on the General Relativity.

Among the various methods that are used to construct exact solutions of equations (1)–(2), the method of bringing one of them to the one-dimensional stationary Schrödinger equation was considered. To our knowledge, the Schrödinger representation of the first Einstein-Friedmann equation (1) was proposed for the first time by Zhuravlev et al [22]. The method was further developed by A. Yurov with coauthors in the works [23, 24]. Later Barbosa-Cendejas and Reyes [25] repeated the derivation of the Schrödinger equation from the Friedmann equation, and compared solutions in cosmology and quantum mechanics. In our recent work [26], another approach was considered, based on the representation of the first Einstein-Friedman equation as the Schrödinger-like equation in terms of a scalar field.

In this paper, we consider the representation of the second Einstein-Friedman equation (2) in the form of the Schrödinger-like equation and give examples of known and new exact cosmological solutions obtained by this method. Also, this approach provides a new way of comparing quantum-mechanical and cosmological problems, as shown by the example of the Pöschl–Teller potential.

Further, we consider the possibility of applying the Darboux transformations within the framework of the proposed approach. It is shown that one can use these transformations both to generate new exact solutions from known ones in models with one scalar field, and in multi-field Chiral Cosmological Models (CCM) [27–32] as well.

Finally, we generalize the representation of both Einstein-Friedman equations as a one-dimensional stationary Schrödinger-like equation, which allows one to compare the solutions obtained by using any other methods to this approach.

## 1. Schrödinger-like representation of second Einstein-Friedmann equation

It is well known that to generate exact solutions of the system (1)–(3) in explicit form, it is sufficient to find solutions of the second Einstein-Friedman equation (2) only.

Now, we consider the one-dimensional Schrödinger-like equation in terms of the cosmic time

$$\ddot{\psi} - u(t)\psi = 0, \tag{4}$$

where $\psi = \psi(t)$ is a some function of time. After the following function change

$$u(t) = \ddot{\phi} - 2\dot{H}, \tag{5}$$

$$\psi(t) = \mu_1 \exp(\phi(t)), \tag{6}$$

where $\mu_1 \neq 0$ is the constant, from equation (4) we obtain the second Einstein-Friedmann equation (2)

$$\dot{\phi}^2 = -2\dot{H}. \tag{7}$$

The inverse transformations of the equations (5)–(6) give

$$\phi(t) = \ln\left(\frac{1}{\mu_1}\psi(t)\right), \tag{8}$$

$$H(t) = \frac{1}{2}\left(\frac{\dot{\psi}}{\psi} - \int u(t)dt + \lambda\right), \tag{9}$$

where the functions $u(t)$ and $\psi(t)$ are connected by equation (4).

Also, from equations (1)–(2) one has the expression for the potential of a scalar field

$$V(\phi(t)) = 3H^2 + \dot{H}. \tag{10}$$

Further, we will consider some exact cosmological solutions for some potentials $u(t)$.

### 1.1. Solutions for $u = 0$

For the case $u(t) = 0$, from equation (4) we obtain

$$\psi(t) = \mu_1(c_1 t + c_2), \tag{11}$$

where $c_1$ and $c_2$ are constants of integration. From (8)–(9) and (10) one has

$$\phi(t) = \ln(c_1 t + c_2), \tag{12}$$

$$H(t) = \frac{c_1\lambda t + c_2\lambda + c_1}{2(c_1 t + c_2)}, \tag{13}$$

$$a(t) = a_0 e^{\frac{1}{2}\lambda t}(c_1 t + c_2)^{1/2}, \tag{14}$$

$$V(\phi) = \frac{c_1^2}{4}e^{-2\phi} + \frac{3c_1\lambda}{2}e^{-\phi} + \frac{3}{4}\lambda^2. \tag{15}$$

These solutions correspond to exponential power-law inflation [7]. For $c_1 = 0$ we have the de Sitter solution with $\phi = \ln(c_2) = const$, $H = \frac{\lambda}{2} = const$ and $V = \frac{3}{4}\lambda^2 = const$ as the partial solution.

### 1.2. Solutions for $u = const \neq 0$

For the case $u(t) = A = const$, from equation (4) we obtain

$$\psi(t) = \mu_1\left(c_1 e^{\sqrt{A}t} + c_2 e^{-\sqrt{A}t}\right), \tag{16}$$

where $c_1$ and $c_2$ are the constants of integration. Now, we note the growing and decaying solutions

$$\psi_{1,2}(t) = \exp\left(\pm\sqrt{A}t + \phi_0\right), \quad \phi_0 = const. \tag{17}$$

From (8)–(9) and (10) we obtain exact solutions for chaotic inflation [5, 19]

$$\phi(t) = \pm\sqrt{A}\,t + \phi_0, \tag{18}$$

$$H(t) = \frac{1}{2}\left(\lambda \pm \sqrt{A} - At\right), \tag{19}$$

$$a(t) = a_0 \exp\left\{\frac{1}{2}\left[\left(\lambda \pm \sqrt{A}\right)t - \frac{At^2}{2}\right]\right\}, \tag{20}$$

$$V(\phi) = \left[\lambda \pm \sqrt{A} \pm \sqrt{A}\,(\phi_0 - \phi)\right]^2 - \frac{1}{2}A. \tag{21}$$

The other wave functions derived from the conditions $c_1 = c_2$, $c_1 = -c_2$ don't lead to physical inflationary potentials.

### 1.3. Solutions for the Pöschl–Teller quantum mechanical potential

Now, we consider the following wave function

$$\psi(t) = \mu_1 \tanh(\alpha t), \tag{22}$$

where $\alpha$ is an arbitrary constant. From equation (4) we obtain the quantum mechanical Pöschl–Teller potential [33]

$$u(t) = -\frac{2\alpha^2}{\cosh^2(\alpha t)}. \tag{23}$$

From (8)–(9) and (10) for $\lambda = 0$ one has the corresponding cosmological model

$$\phi(t) = \ln(\tanh(\alpha t)), \tag{24}$$

$$H(t) = \alpha \cot(2\alpha t), \tag{25}$$

$$a(t) = a_0[\sinh(2\alpha t)]^{1/2}, \tag{26}$$

$$V(\phi) = \alpha^2[\cosh^2(\phi) + 2]. \tag{27}$$

Similar solutions for the potential (27) were considered earlier in [11, 13]. Hence, we have a connection between the cosmological and quantum mechanical problems for the case considered.

### 1.4. Generalization of inflationary models with polynomial potentials for the small scalar field

Now, we consider the wave function

$$\psi(t) = \mu_1 \exp\left\{\frac{1}{C}\arcsin\left[\exp\left(-2AC^2(t+c_1)\right)\right]\right\}, \tag{28}$$

where $A$ and $C$ are arbitrary constants. From equations (4)–(10) we obtain the exact solutions

$$H(t) = \frac{A}{2}\ln\left[1+\exp\left(-4AC^2(t+c_1)\right)\right] + B, \tag{29}$$

$$\phi(t) = \frac{1}{C}\arcsin\left[\exp\left(-2AC^2(t+c_1)\right)\right], \tag{30}$$

$$a(t) = a_0 \exp\left(\frac{1}{8C^2}\left\{8BC^2 t + f\left[1+\exp(-4AC^2(t+c_1))\right]\right\}\right), \tag{31}$$

$$V(\phi) = 3\left(A\ln[\cosh(C\phi)] + B\right)^2 - 2A^2C^2\tanh^2(C\phi), \tag{32}$$

where $B$ is the constant of integration and the function $f(\xi)$ is defined as

$$f(\xi) = \int_1^{\xi} \frac{\ln(\xi)}{1-\xi} d\xi. \tag{33}$$

For the small scalar field $\phi \ll 1$ from (32) we obtain the double-well potential

$$V(\phi) = \left(-\frac{1}{2}ABC^4 + \frac{3}{4}A^2C^4 + \frac{4}{3}A^2C^6\right)\phi^4 + \left(-2A^2C^4 + 3ABC^2\right)\phi^2 + 3B^2 + \mathcal{O}(\phi^6), \tag{34}$$

Therefore, for different choices of the constants $A$, $B$ and $C$ we have the different potentials as the partial cases. For the case $B = \frac{2}{3}AC^2$ we have

$$V(\phi) = A^2C^4\left(C^2 + \frac{3}{4}\right)\phi^4 + \mathcal{O}(\phi^6), \tag{35}$$

For the case $B = \frac{8}{3}AC^2 + \frac{3}{2}A$ we obtain

$$V(\phi) = 3AC^2\left(2C^2 + \frac{3}{2}\right)\phi^2 + \mathcal{O}(\phi^6), \tag{36}$$

The case $B = -\frac{1}{2}A$ and $C = \pm\frac{\sqrt{3}i}{2}$ corresponds to the potential

$$V(\phi) = 3B^2 + \mathcal{O}(\phi^6). \tag{37}$$

The evolution of the remaining parameters of these models is determined by substitution of the constants in solutions (29)–(31). Thus, we have new cosmological solutions for known potentials which are considered in [3–5] with negligible corrections for the small scalar field.

## 2. Darboux class of exact cosmological solutions

One of the possible form-invariant transformations of the one-dimensional stationary Schrödinger equation is the Darboux transformations [34–36]. It should be noted that the application of such a transformations to the first Einstein-Friedmann equation, written in different forms, was discussed in [23, 26]. In this case, we will consider the application of the Darboux transformations to the second Einstein-Friedman equation to generate new exact cosmological solutions from known ones and for conversion of exact solutions from the case of single-field models to multi-field Chiral Cosmological Models (CCM) as well.

### 2.1. Single field cosmological models

Now, we consider the one-dimensional Schrödinger equation in terms of the cosmic time

$$\ddot{\tilde{\psi}} - \tilde{u}(t)\tilde{\psi} = 0, \tag{38}$$

where $\tilde{\psi} = \tilde{\psi}(t)$ is a some function of time. After the following function change

$$\tilde{u}(t) = \ddot{\varphi} - 2\dot{\tilde{H}}, \tag{39}$$

$$\tilde{\psi}(t) = \mu_2 \exp(\varphi(t)), \tag{40}$$

where $\mu_2 \neq 0$ is the constant, from equation (4) we obtain the second Einstein-Friedmann equation (2)

$$\dot{\varphi}^2 = -2\dot{\tilde{H}}. \tag{41}$$

We will consider $\psi$ and $\tilde{\psi}$ as a partial solutions of the equations (4) and (38). The connection between this solutions can be obtained from the Darboux transformations

$$\tilde{u} = u - 2\frac{d^2}{dt^2}\ln(f(t)), \tag{42}$$

$$\tilde{\psi} = \dot{\psi} - \psi\left\{\frac{d}{dt}\ln(f(t))\right\}, \tag{43}$$

where $f(t)$ is the general solution of the equation (4)

$$\ddot{f} - u(t)f = 0. \tag{44}$$

Therefore, based on these transformations, one can obtain the connection between the exact solutions of the equation (7) and (41) in the following form

$$\varphi(t) = \sqrt{n}\,[\phi(t) + \chi(t)] + \varphi_0, \tag{45}$$

$$\tilde{H}(t) = n\left[H(t) + \frac{\dot{f}}{f} + \frac{1}{2}\dot{\chi}\right] + \lambda, \tag{46}$$

$$\chi(t) = \ln\left[\frac{\mu_1}{\mu_2}\left(\dot{\phi} - \frac{\dot{f}}{f}\right)\right], \tag{47}$$

$$\ddot{f} - \left(\ddot{\phi} - 2\dot{H}\right)f = 0, \tag{48}$$

where $n$, $\lambda$ and $\varphi_0$ are some constants.

The general solution $f = \psi^{(1)} + \psi^{(2)}$ of the equation (48) can be found on the basis of the expression for $u = \ddot{\phi} - 2\dot{H}$ and known particular solution $\psi^{(1)}(t) = \mu_1 \exp(\phi(t))$. For the case $\frac{\mu_1}{\mu_2}\left(\dot{\phi} - \frac{\dot{f}}{f}\right) > 0$ one has a canonical field $\varphi(t)$ for $n > 0$ and phantom one for $n < 0$. For $\frac{\mu_1}{\mu_2}\left(\dot{\phi} - \frac{\dot{f}}{f}\right) < 0$ we have a complex scalar field $\varphi(t)$ in which real and imaginary components depend on the sign of $n$.

Also, one can define the new potential as

$$\tilde{V}(\varphi(t)) = 3\tilde{H}^2 + \dot{\tilde{H}}. \tag{49}$$

Thus, from known solutions $\phi$ and $H$ of equation (7) one can obtain the new exact solutions $\varphi$, $\tilde{H}$ and $\tilde{V}$ from expressions (45)–(49).

### 2.2. Two field cosmological models

As one can see, the function $\chi(t)$ can be considered as the additional scalar field. After substituting the scalar field

$$\varphi(t) = \sqrt{n}\,[\phi(t) + \chi(t)] + \varphi_0 \tag{50}$$

into the Einsten-Friedmann equations (1)–(2) we obtain

$$3\tilde{H}^2 = \frac{n}{2}\dot{\phi}^2 + n\dot{\phi}\dot{\chi} + \frac{n}{2}\dot{\chi}^2 + V(\phi,\chi), \tag{51}$$

$$-\dot{\tilde{H}} = \frac{n}{2}\dot{\phi}^2 + n\dot{\phi}\dot{\chi} + \frac{n}{2}\dot{\chi}^2. \tag{52}$$

After substituting the field (50) into the field equation (3) one has

$$\begin{aligned}\sqrt{n}(\ddot{\phi} + \ddot{\chi}) + 3\sqrt{n}\tilde{H}(\dot{\phi} + \dot{\chi}) &= -\frac{dV(\varphi)}{d\varphi} = \frac{dV}{dt}\frac{dt}{d\varphi} = \\ &= -\left(\frac{\partial V(\phi,\chi)}{\partial \phi}\dot{\phi} + \frac{\partial V(\phi,\chi)}{\partial \chi}\dot{\chi}\right)\frac{1}{\sqrt{n}(\dot{\phi} + \dot{\chi})}.\end{aligned} \tag{53}$$

$$n(\ddot{\phi} + \ddot{\chi})(\dot{\phi} + \dot{\chi}) + 3n\tilde{H}(\dot{\phi} + \dot{\chi})^2 = -\frac{\partial V(\phi,\chi)}{\partial \phi}\dot{\phi} - \frac{\partial V(\phi,\chi)}{\partial \chi}\dot{\chi}. \tag{54}$$

$$\begin{aligned}&\ddot{\phi}\dot{\phi} + \ddot{\chi}\dot{\phi} + 3\tilde{H}(\dot{\phi}^2 + \dot{\chi}\dot{\phi}) + \ddot{\phi}\dot{\chi} + \ddot{\chi}\dot{\chi} + 3\tilde{H}(\dot{\chi}^2 + \dot{\phi}\dot{\chi}) = \\ &= -\frac{1}{n}\left(\frac{\partial V(\phi,\chi)}{\partial \phi}\dot{\phi} + \frac{\partial V(\phi,\chi)}{\partial \chi}\dot{\chi}\right).\end{aligned} \tag{55}$$

Therefore, the field equation (55) can be represented as two ones in the following form

$$\ddot{\phi} + 3\tilde{H}(\dot{\phi} + \dot{\chi}) + \ddot{\chi} = -\frac{1}{n}\frac{\partial V(\phi,\chi)}{\partial \phi}, \tag{56}$$

$$\ddot{\chi} + 3\tilde{H}(\dot{\phi} + \dot{\chi}) + \ddot{\phi} = -\frac{1}{n}\frac{\partial V(\phi,\chi)}{\partial \chi}. \tag{57}$$

Such a model with dynamic equations (51)–(52) and (56)–(57) containing a mixed kinetic terms was considered earlier in the paper [37] for $n = 1$.

Also, we note, that this system has two independent equations only, because the equation (53) and (56)–(57) can be obtained from equations (51)–(52).

Thus, based on the transformations (45)–(48) one can generate the exact cosmological solutions for this system of equations from known $\phi$ and $H$ following from (7).

### 2.3. Multi-field cosmological models

The previous models with mixed kinetic terms is the partial case of the chiral cosmological models (CCM) with $K$ scalar fields $\phi^A$ ($\tilde{\phi} = \phi^0, \phi^1, \phi^2..., \phi^K$) ($K = 2$) based on the action [27–32]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}h_{AB}\partial_\mu\phi^A\partial_\nu\phi^B g^{\mu\nu} - V(\tilde{\phi})\right], \tag{58}$$

where the $h_{AB} = n\hat{I}$, and $\hat{I}$ is the unit matrix.

For the CCM with K-fields, in the spatially flat Friedmann–Robertson–Walker metric, from the action (58), one has the following dynamic equations

$$3\tilde{H}^2 = \frac{1}{2}h_{AB}\dot{\phi}^A\dot{\phi}^B + V(\tilde{\phi}), \tag{59}$$

$$-\dot{\tilde{H}} = \frac{1}{2}h_{AB}\dot{\phi}^A\dot{\phi}^B, \tag{60}$$

$$h_{CB}(\ddot{\phi}^B + 3\tilde{H}\dot{\phi}^B) + V_{,C} = 0. \tag{61}$$

We start from the equation for one scalar field

$$-2\dot{H}^0 = \dot{\phi}^0\dot{\phi}^0 = \left(\dot{\phi}^0\right)^2. \tag{62}$$

The first way to construct the exact solutions from known $\phi^0$ and $H^0$ is to represent the field $\chi$ as the sum of the other fields $\chi = \phi^1 + \phi^2 + ...\phi^N$.

Thus, from the transformations (45)–(48), one has

$$\tilde{\phi}(t) = \sqrt{n}\left[\phi^0(t) + \sum_{B=1}^{N}\phi^B\right] + const, \tag{63}$$

$$\tilde{H}(t) = n\left[H^0(t) + \frac{\dot{f}}{f} + \frac{1}{2}\left(\frac{d}{dt}\sum_{B=1}^{N}\phi^B\right)\right] + \lambda, \tag{64}$$

$$\sum_{B=1}^{N}\phi^B = \ln\left[\frac{\mu_0}{\mu_1}\left(\dot{\phi}^0 - \frac{\dot{f}}{f}\right)\right], \tag{65}$$

$$\ddot{f} - \left(\ddot{\phi}^0 - 2\dot{H}^0\right)f = 0, \tag{66}$$

$$V(\tilde{\phi}(t)) = 3\tilde{H}^2 + \dot{\tilde{H}}, \tag{67}$$

where one can consider any scalar fields $\phi^B$ corresponding to the condition (65).

The second way is to use the Darboux transformation $K$-times. Each Darboux transformation of the equation (60) gives one additional field, therefore one has the following equations

$$\tilde{\phi}(t) = \sqrt{n}\sum_{A=0}^{K}\left[\phi^A(t) + \phi^{A+1}(t)\right] + const, \tag{68}$$

$$\tilde{H}(t) = n\sum_{A=0}^{K}\left[H^A(t) + \frac{\dot{f}^A}{f^A} + \frac{1}{2}\dot{\phi}^{A+1}\right] + \lambda, \tag{69}$$

$$\phi^{A+1}(t) = \ln\left[\frac{\mu_A}{\mu_{A+1}}\left(\dot{\phi}^A - \frac{\dot{f}^A}{f^A}\right)\right], \tag{70}$$

$$\ddot{f}^A - \left(\ddot{\phi}^A - 2\dot{H}^A\right)f^A = 0, \tag{71}$$

$$V(\tilde{\phi}(t)) = 3\tilde{H}^2 + \dot{\tilde{H}}. \tag{72}$$

One can also combine these approaches to construct exact cosmological solutions for multi-field Chiral Cosmological Models.

### 3. Generalized Schrödinger-like representation of cosmological dynamic equations

Now, we combine the method under consideration and the other approach which was considered earlier in [26]. The basis of this approach is representation of a first Einstein-Friedmann as Schrödinger-like one with corresponding dynamic equations (1)–(3) in following form [26]

$$\left[-\frac{d^2}{d\phi^2}+U(\phi)\right]\psi(\phi)=0, \tag{73}$$

$$V'_\phi=6\left[1-\frac{2}{3}U(\phi)\right]\psi\psi'_\phi, \tag{74}$$

$$\dot{\phi}=-2\psi'_\phi, \tag{75}$$

where $\psi(\phi)\equiv H(\phi)$, therefore, in this case, the Hubble parameter playing role a wave function in equation (73).

Thus, on the basis of equations (4)–(9) and (73)–(75) we can conclude that for cosmological inflationary models containing a scalar field and based on Einstein gravity in a flat four-dimensional Friedmann-Robertson-Walker space-time, the exact solutions of the system of dynamical equations (1)–(3) obtained by using any methods, can also be obtained based on the Schrödinger-like equation

$$\frac{d^2\psi(x)}{dx^2}-U(x)\psi(x)=0, \tag{76}$$

for which the case $x\equiv\phi$, $U(x)=U(\phi)$ corresponds to the relations

$$V'_\phi=6\left[1-\frac{2}{3}U(\phi)\right]\psi\psi'_\phi, \tag{77}$$

$$\dot{\phi}=-2\psi'_\phi, \qquad H(\phi)=\psi(\phi), \tag{78}$$

and the case $x\equiv t$, $U(x)=u(t)$ correspond to the relations

$$\dot{H}=\frac{1}{2}\left[\frac{d}{dt}\left(\frac{\dot{\psi}}{\psi}\right)-u(t)\right], \tag{79}$$

$$\phi(t)=\ln(\psi(t)), \qquad V(\phi(t))=3H^2+\dot{H}. \tag{80}$$

It should also be noted that some solutions of equation (76) correspond to different solutions of equations (77)–(78) and (79)–(80).

Thus, one can investigate the exactly solvable cosmological models on the basis of the Schrödinger-type equation only with additional relations between the parameters of the models. This approach gives two alternative ways to connect the quantum mechanical and cosmological problems as well.

We also note, that based on the results presented in [38–43], one can use the proposed approach for constructing exact solutions for cosmological inflationary models with modified gravity theories, namely, with Einstein-Gauss-Bonnet gravity and scalar-tensor gravity as well by the functional and parametric connections between these types of gravity theories and General Relativity in Friedmann universe.

### Conclusion

In this paper we considered an application of the Schrödinger-type equation to construction exact cosmological solutions in inflationary models with scalar field based on General Relativity. The first step in this analysis was a new representation of the second Einstein-Friedmann equation as a one-dimensional stationary Schrödinger-type equation. This representation made it possible to obtain exact cosmological solutions in explicit form. Also, this approach allows us to compare quantum-mechanical and cosmological problems in a new way.

The second step was to use the Darboux transformations to generate new exact solutions from the known ones. It was also shown that these transformations allow the transition from models with one scalar field to Chiral Cosmological Models with several fields. This approach differs from that proposed in work [44], in which such a transition was carried out due to the specific choice of the target space metric.

Finally, we generalized the representation of both Einstein-Friedman equations as the Schrödinger equation with different conditions for various variables (scalar field or cosmic time). Such a representation of background dynamics equations (1)–(3) is quite convenient since, on the one hand, any exact solutions for an unperturbed scalar field can be obtained in the presented way, on the other hand, the evolution equations of scalar $v_k$ and tensor $u_k$ cosmological perturbations in linear order of perturbations theory are also can be written as the Schrödinger-type equations, namely [45, 46]

$$\frac{d^2 v_k}{d\eta^2} + \left(k^2 - \frac{1}{z}\frac{d^2 z}{d\eta^2}\right) v_k = 0, \tag{81}$$

$$\frac{d^2 u_k}{d\eta^2} + \left(k^2 - \frac{1}{a}\frac{d^2 a}{d\eta^2}\right) u_k = 0, \tag{82}$$

where $z = a\dot{\phi}/H$, $k$ is the wave number and $\eta = \int dt/a$ is the conformal time.

Thus, the same task, from a mathematical point of view, corresponds to two different levels of analysis of cosmological models that leads to the assertion that the whole problem of constructing models of the early universe with a scalar field on the basis of General Relativity can be reduced to an analyzing of same type equations (76) and (81)–(82).

The prospect of using an approach based on the application of the Schrödinger equation to the analysis of cosmological models consists in developing existing and constructing new effective methods for exact and approximate solutions of this type of equation or developing effective algorithms for its numerical solutions, which will allow to comprehensively solve the problem of constructing verifiable models of the early universe corresponding to observational constraints on the parameters of cosmological perturbations [47]. The development of this approach is the task of our following investigations in this direction.

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### Авторы

**Фомин Игорь Владимирович**, Московский государственный технический университет имени Н.Э. Баумана, 2-я Бауманская ул., д. 5, г. Москва, 105005, Россия.
E-mail: ingvor@inbox.ru

**Червон Сергей Викторович**, профессор, д. ф.-м. н., кафедра физики и технических дисциплин, Ульяновский государственный педогогический университет, пл. Ленина, д. 4/5, г. Ульяновск, 432071, Россия; профессор кафедры физики, Московский государственный технический университет имени Н.Э. Баумана, 2-я Бауманская ул., д. 5, г. Москва, 105005, Россия; ведущий научный сотрудник Казанского федерального университета, ул. Кремлевская, д. 18, г. Казань, 420008, Россия.
E-mail: chervon.sergey@gmail.com

**Махарадж Сунил Датт**, профессор, д. ф.-м. н., отдел исследований астрофизики и космологии, школа математики, статистики и компьютерных наук, Университет Квазулу–Натал, X54001, г. Дурбан, 4000, Южная Африка.
E-mail: maharaj@ukzn.ac.za

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### Authors

**Fomin Igor Vladimirovich**, Doctor of Physical and Mathematical Sciences, Professor, Physics Department, Bauman Moscow State Technical University, 2-nd Baumanskaya st., 5, Moscow, 105005, Russia.

E-mail: ingvor@inbox.ru

**Chervon Sergey Viktorovich**, Doctor of Physical and Mathematical Sciences, Professor, Department of Physics and Technical Disciplines, Ulyanovsk State Pedagogical University, Lenin's square, Build. 4/5, Ulyanovsk, 432071, Russia; Professor of Physics Department of Bauman Moscow State Technical University, 2-nd Baumanskaya st., 5, Moscow, 105005, Russia.
E-mail: chervon.sergey@gmail.com

**Maharaj Sunil Dutt**, Doctor of Physical and Mathematical Sciences, Professor, Astrophysics and Cosmology Research Unit, School of Mathematics, Statistics and Computer Science, University of KwaZulu–Natal, Private Bag X54001, Durban 4000, South Africa.
E-mail: maharaj@ukzn.ac.za

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