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Thermodynamics of Cosmological Models with a Variable Matter Equation of State

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Abstract—We consider cosmological models with a self-interacting scalar field and a perfect fluid with a variable equation of state in spatially flat Friedmann-Robertson-Walker space-times. The main purpose of the paper is a study of general thermodynamic properties of such models and their relationship with the dynamics of geometry.

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1. INTRODUCTION

At present, there is great interest in cosmological models with a variable matter equation of state in the class of equations $p = \gamma(t)\varepsilon$ (p is the fluid pressure and ε its energy density). By now, methods allowing for restoration of the quantity $\gamma(t)$ from expressional data have been developed [1], and an analysis of the experimental data has been conducted to determine this parameter as a function of cosmological time (see [2] and references therein). However, in all cited papers, the parameter $\gamma(t)$ is calculated with some reasoning which does not have a thermodynamic nature but is rather reduced to some simple parametrizations of the dependences $\gamma = \gamma(t)$, as is done, for instance, in [3–8]. On the other hand, the thermodynamic meaning of the parameter $\gamma(t)$ does require a thermodynamic approach for a description of its dynamics. The thermodynamic approach to the description of dark energy and the related properties of matter is at present under active discussion. The thermodynamic properties of dark energy in the form of a perfect fluid have been recently analyzed [9] for different values of $\gamma = \text{const}$. Some thermodynamic aspects of models with $\gamma = \gamma(t)$ have been studied in [10–12]. Actually, a thermodynamic approach to model construction with $\gamma = \gamma(t)$ has been developed in [14] on the basis of a special representation of the Einstein equations worked out in [13] for a spatially flat Universe.

In [14], a two-component cosmological model was studied, with a Friedmann-Robertson-Walker (FRW) spatially flat metric and matter in the form of a scalar field and a perfect fluid with a variable equation of state of the general form $p = \gamma(t)\varepsilon$. The main idea was to obtain such a scenario of the Universe development, beginning with an inflationary stage, where usual matter could emerge as a result of evolution from a scalar field and quasivacuum which filled the Universe from the very beginning. A transition of some part of matter from a quintessence-like state or dark energy to its usual form could be related to a thermal equilibrium of the two forms of matter, preserved at all stages. This would allow finding an equation for the parameter $\gamma(t)$ in an explicit form from the equality requirement for temperatures corresponding to the field and material components. An analysis was conducted for a whole class of such models, distinguished by a general condition on the evolution of the total energy density $\mathcal{W}(\phi(t))$ of the scalar field ϕ:

$$\dot{\mathcal{W}} = -k\mathcal{W}^{\alpha}, \qquad (1)$$

where k and α are constants. In all such models with $1 < \alpha \leq 3/2$, the total field energy decreases by a power law while the field self-interaction potential, if matter is absent (i.e., for $\varepsilon \equiv 0$), has a form similar to the Higgs potential and is determined by the relation $V(\phi) = A\phi^M - B\phi^N$ with the exponents $M, N > 2$ (see [14]). The instant $t = 0$ in such models corresponds to a cosmological singularity. As was shown, in all such models the parameter $\gamma(t)$ evolves in such a way that $\lim_{t\to 0}\gamma = -1$ and $\lim_{t\to\infty}\gamma = -1$. That means that always, irrespective of the initial conditions, the Universe sooner or later should begin to expand with acceleration while matter acquires the properties of a quasivacuum (dark energy). In addition, such models inevitably contain an epoch dominated by matter with the parameter $\gamma \geq 0$. The importance of this conclusion is that, on the one hand, it is this situation that corresponds to the existing picture of the Universe evolution up to the present epoch; on the other hand, this result need a minimum of material and mathematical ingredients for its foundation. This makes such a class of models promising enough for creating models which would also incorporate the evolution of density perturbations. Models with a variable equation of state at present seem to be most adequate to be confronted to the experimental data. A number of papers are devoted to analyzing methods of verifying such models with the experimental data and verification itself, see, e.g., [1, 2]. Therefore, in the present paper, the class of models suggested previously is further generalized and modified to obtain a more adequate representation of the real Universe evolution.

We will here analyze a more general class of models for which the dynamics is determined in other ways as compared to [14], i.e., based on the assumption (1). We will suggest a method of specifying the evolutionary characteristics by certain relations involving the field ϕ or its total energy, to be called *the model with a master scalar field*. We will give it in a more general form and supply it with a more reliable foundation than that given in [14]. This foundation will rest on studying general thermodynamic consequences of the equilibrium conjecture for the state of matter at all stages of the Universe evolution. From this analysis, we will derive other kinds of models able to be considered in the framework of the present approach.

2. BASIC ELEMENTS OF FIELD DYNAMICS

A starting point to an analysis of the models to be considered are the standard Einstein equations for a spatially flat FRW metric with a self-interacting scalar field ϕ and a perfect fluid:

$$H^2 = \frac{\kappa}{3}\left(\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] + \varepsilon\right), \qquad (2)$$

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{d}{d\phi}V(\phi). \qquad (3)$$

Here, $H = \dot{R}/R$ is the Hubble parameter, R is the scale factor, κ is the Einstein gravitational constant, ε is the fluid energy density, and $V(\phi)$ is the self-interaction potential of the field ϕ. The first is just the Einstein equation while the second one is the ϕ field equation. This set of equations is a basis for analyzing the majority of cosmological scenarios in the framework of the FRW metric [15].

The above equations, in the analysis of inflationary scenarios, are usually simplified by the slow-rolling approximation [15]. It has been shown [13], however, that, by introducing the total field energy potential, Eqs. (2) and (3) are in a simple way converted to

$$H^2 = \frac{\kappa}{3}\Big(W(\phi) + \varepsilon\Big), \qquad (4)$$

$$3H\dot{\phi} = -\frac{d}{d\phi}W(\phi), \qquad (5)$$

where, insted of the self-interaction potential $V(\phi)$, one uses the total scalar field energy $W(\phi)$:

$$W(\phi) = V(\phi) + \frac{1}{2}\dot{\phi}^2(\phi) = V(\phi) + \frac{1}{2}U^2(\phi). \qquad (6)$$

Here, we have denoted

$$\dot{\phi} = U(\phi). \qquad (7)$$

Since the model contains two components of matter, the field and the fluid, one should supplement the above equations with an equation for the fluid pressure p:

$$p = -\frac{1}{æ}\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) - \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad (8)$$

which is, generally speaking, a consequence of the previous ones. This equation makes it possible to convert the above equations to a more convenient form using some simple transformations [13, 14]:

$$\mathcal{P} = -W - \frac{1}{\sqrt{3\kappa}}\frac{\dot{W}}{\sqrt{W+\varepsilon}}, \qquad (9)$$

$$p = -\varepsilon - \frac{1}{\sqrt{3\kappa}}\frac{\dot{\varepsilon}}{\sqrt{W+\varepsilon}}, \qquad (10)$$

$$R = R_0 \exp\left\{\sqrt{\kappa/3}\int\sqrt{W+\varepsilon}dt\right\}. \qquad (11)$$

where the effective field pressure $\mathcal{P}$ has the standard form

$$\mathcal{P} = \frac{1}{2}U^2(\phi) - V(\phi). \qquad (12)$$

A useful relation is also the one connecting the scalar field and fluid parameters [13, 14]:

$$\sqrt{3\kappa}U\sqrt{W(\varphi)+\varepsilon} = -W'. \qquad (13)$$

The importance of the representation constructed here for the original equations (2), (3) (without making any assumptions) is that Eqs. (4), (5) precisely coincide with the slow-rolling equations, with the only difference that it contains the total field energy $W(\phi)$ instead of the self-interaction potential $V(\phi)$. It means that all well-known conclusions concerning the cosmological inflation theory may be attributed to the exact equations (4), (5) with the only difference that, instead of $V(\phi)$, the conclusions should be formulated in terms of $W(\phi)$.

3. BASIC ELEMENTS OF THE MODEL THERMODYNAMICS

Following [14], let us also write out the thermodynamic relations to be used in what follows. Concerning the non-field component of matter (i.e., fluid) with the energy density ε, we will suppose that it evolves in such a way that the following equation of state is valid:

$$p = \gamma(t)\varepsilon, \tag{14}$$

where p is the fluid pressure while the parameter $\gamma = \gamma(t)$ is a function of time. Such an equation of state corresponds in a general form [14] to an effective equation of state of a mixture of different components of matter (dust, electromagnetic radiation etc.) with different partial equations of state of the form

$$p_j = \gamma_j\varepsilon_j, \quad j = 1, 2, \ldots$$

For the total pressure $p = \sum_j p_j$ of such a mixture we have

$$p = \sum_j p_j = \sum_j \gamma_j\varepsilon_j = \frac{\sum_j \gamma_j\varepsilon_j}{\sum_j \varepsilon_j}\varepsilon,$$

where ε is the total energy of the mixture: $\varepsilon = \sum_j \varepsilon_j$. Comparing this relation with (14), we find

$$\gamma(t) = \sum_j \gamma_j\varepsilon_j \Big/ \sum_j \varepsilon_j.$$

In particular, for a mixture of dust and electromagnetic radiation.

$$\gamma(t) = \frac{1}{3}\frac{\varepsilon_e(t)}{\varepsilon_e(t) + \varepsilon_d(t)},$$

where ε_e is the energy density of the isotropic radiation and ε_d is that of dust.

As in [14], we will suppose that the parameters of the fluid, considered as a material thermodynamic object, satisfy the standard implications of the second law of thermodynamics for equilibrium processes:

$$\varepsilon = -p + T\frac{\partial p}{\partial T}, \tag{15}$$

where T is the absolute temperature of the system in thermodynamic (thermal) equilibrium. This equation follows from the requirement that for equilibrium processes the matter entropy S should be a function of the system state parameters, in our case, the temperature T and the volume V: $S = S(T, V)$. In this case, Eq. (15) is obtained from the first law of thermodynamics:

$$TdS = d\mathcal{U} + pdV. \tag{16}$$

Here, $\mathcal{U}$ is the internal energy of the system. In principle, one could consider, by analogy with [9], a more general case of a system with a variable particle number, but then it would be necessary to supplement the theory with considerations on how particle creation and destruction take place, thus substantially modifying without any hope to obtain equally substantial results. Therefore we here do not consider such an extension.

The relation (16) leads to the following expressions for the derivatives of the entropy with respect to T and V:

$$\frac{\partial S}{\partial T} = \frac{1}{T}\frac{\partial \mathcal{U}}{\partial T}, \quad \sigma = \frac{\partial S}{\partial V} = \frac{1}{T}\left[\frac{\partial \mathcal{U}}{\partial V} + p\right]. \tag{17}$$

A consistency condition for this system is just given by Eq. (15) where, by definition, one puts

$$\varepsilon = \frac{\partial \mathcal{U}}{\partial V}.$$

The second equation in (17) gives a relationship between the entropy density σ and the energy density:

$$\sigma = \frac{\varepsilon + p}{T} = (1+\gamma)\frac{\varepsilon}{T}. \tag{18}$$

Excluding the pressure p from Eq. (15) with the aid of (10), we obtain:

$$\frac{d\ln T}{dt} = \sqrt{3æ}\sqrt{W+\varepsilon} + \frac{d}{dt}\ln\left[\frac{\dot\varepsilon}{\sqrt{3æ}\sqrt{W+\varepsilon}}\right].$$

Using (11), this relation is reduced to a full time derivative, which results in the entropy conservation law which we write down in the form

$$T = CR^3\frac{\dot\varepsilon}{\sqrt{3æ}\sqrt{W+\varepsilon}}, \tag{19}$$

where C is an integration constant. Using again (9), (10), we can write the latter relation as an entropy conservation law in an elementary volume of matter in a standard form:

$$R^3(1+\gamma)\frac{\varepsilon}{T} \equiv R^3\sigma = -\frac{1}{C} = s_0 = \text{const}. \tag{20}$$

Here, s_0 is the entropy of a comoving fluid volume.

From (19) one can make an important conclusion: since, by definition, $T > 0$, $R > 0$ and $s_0 > 0$, it follows from (20) that, during the whole Universe evolution in an equilibrium regime, the inequality $\dot\varepsilon \leq 0$ holds, i.e., the energy density of the material component monotonically decreases. The energy density can grow only if the thermodynamic equilibrium of the system is violated.

4. MODELS WITH A MASTER FIELD

Let us note that the system (9)–(11) contains an unspecified function. It is connected with the fact that the self-interaction potential of the scalar field is actually unknown. This inserts arbitrariness into the theory, and it is generally excluded using different additional assumptions on the nature of evolution of the ϕ field itself, the scale factor R or other parameters of the system, which makes a basis for different variants of the so-called potential fine tuning method. One of such approaches is the one developed in [13, 14] (see also references therein). Other possible ways of excluding the uncertainty in the system (9)–(11) reduce to different ways of calculating the potential from some other physical theories which also contain a number of uncertainties of another nature.

Following [13, 14], let us consider a new variant of the potential fine tuning method, to be called the master field model. Master field models rest on two hypotheses. The first of them consists in assuming that the scalar field evolution is not connected locally with changes in the scale factor an matter parameters but is determined by internal causes of evolution of the field itself. Such an internal mechanism could be, for instance, quantum dynamics of the field, e.g., spontaneous decay of its quanta which, on the average, should be described by a simple equation characterizing a change in its total energy density W. The second hypothesis, concerning the field thermodynamics, is formulated as follows. We note that the field itself may be considered as an equilibrium thermodynamic object. In other words, we can introduce such a parameter Θ that the field parameters, its effective pressure $\mathcal{P}$ and total energy W are connected by a relation similar to (15):

$$W = -\mathcal{P} + \Theta\frac{\partial \mathcal{P}}{\partial \Theta}. \qquad (21)$$

The parameter Θ plays the role of an effective temperature of the scalar field as a thermodynamic object. This relation may be simply treated as a definition of Θ. It can also be considered as an implication of an analogue of the first law of thermodynamics (see the Appendix) of the form

$$\Theta d\mathcal{S} = d\mathcal{W} + \mathcal{P}dV,$$

where $\mathcal{S}$ is the scalar field entropy. As in the case of a perfect fluid, using Eqs. (9), (11), (21) and excluding the effective pressure $\mathcal{P}$, we obtain the scalar field entropy conservation law

$$R^3\mathcal{S} = \mathcal{S}_0 = \text{const}. \qquad (22)$$

This law may be considered as an integral of motion for the scalar field in such models with two-component matter. This law implies an expression foe Θ similar to (19):

$$\Theta = -\frac{1}{\mathcal{S}_0}R^3\frac{\dot{W}}{\sqrt{3æ}\sqrt{W+\varepsilon}}. \qquad (23)$$

Here, $\mathcal{S}_0$ is the conserved thermodynamic entropy of the field ϕ. And since by the definition of the usual scalar field

$$\mathcal{P} + W = (\dot{\phi})^2 > 0,$$

it immediately follows from (9) that $\dot{W} < 0$. This inequality allows us to conclude from (23) that if the constant $\mathcal{S}_0$ is chosen to be positive, then automatically for all t we have $\Theta \geq 0$. Thus the variable Θ is endowed with all properties of temperature (see the Appendix). In this connection, the second hypothesis consists in postulating that the scalar field temperature Θ and the temperature of matter, T, are equal: $\Theta = T$, i.e., they are in effective temperature equilibrium with each other.

Unifying these hypotheses, we obtain a theory where the scalar field ϕ, evolving according to its internal laws, governs the whole system through thermal equilibrium and thus ultimately governs dynamics of the Universe expansion. Such a concept has been considered in [14]. In the most general form, the internal evolution equation for the scalar field may be represented as follows:

$$\dot{W} = -Q(W) \leq 0, \qquad (24)$$

where $Q(W)$ is a certain nonnegative function. The form of this function determines different kinds of models with a master field. In [14] only power laws, $Q(W) = kW^\alpha \geq 0$ with $1 < \alpha \leq 3/2$, have been considered.

Construction of the dynamics in this approach reduces to the following. From the temperature equality condition, comparing (19) and (23), we arrive at the simple relation

$$\dot{\varepsilon} = \dot{W}, \qquad (25)$$

whence

$$W(t) = \varepsilon(t) - \varepsilon_\infty, \qquad (26)$$

where ε_∞ is an integration constant equal to the value of $\varepsilon(t)$ as $t \to \infty$. Then, due to the condition $T \geq 0$, the functions $\varepsilon(t)$ and $W(t)$ should monotonically decrease. If now the fluid has the equation of state (14), then for the parameter $\delta(t) = \gamma(t) + 1$ from (10) we obtain a simple relation useful for calculations:

$$\delta = \gamma + 1 = -\frac{1}{\sqrt{3\kappa}}\frac{\dot{W}}{(\varepsilon_\infty + W)\sqrt{2W+\varepsilon_\infty}}. \qquad (27)$$

For experimental data analysis [1, 2, 10], it is convenient to represent the parameter δ as a function of the Hubble parameter H. To do so, one can use Eq. (2). With (24), the general relation for δ is obtained in the following form:

$$\delta = \gamma + 1 = \frac{2\kappa}{3}\frac{Q\left(3H^2/\kappa\right)}{(3H^2 - \varepsilon_\infty\kappa)H}. \qquad (28)$$

Since we have assumed in the present model that the ϕ field is evolving according to its own internal laws, the function $Q(W)$ and consequently $W(t)$ as a solution to (24) should be known by the moment when the parameter γ is calculated according to (27). In this case we can describe the matter evolution completely. In particular, from the requirement that the total field energy is positive and is monotonically decreasing it follows that, as $t \to \infty$, for all such models

$$\lim_{t\to\infty} W = W_\infty \geq 0, \quad \lim_{t\to\infty} \dot{W} \to 0,$$

and consequently if in the model $\varepsilon_\infty + 2W_\infty > 0$, then

$$\lim_{t\to\infty} \delta = 0, \quad \lim_{t\to\infty} \gamma = -1.$$

Thus in all such models the Universe inevitably reaches a de Sitter stage of accelerated expansion. Then, there are three main variants of such accelerated expansion. The first one

$$W_\infty > 0, \qquad \varepsilon_\infty = 0,$$

corresponds to the definition of quintessence [16].

The second one,

$$W_\infty = 0, \qquad \varepsilon_\infty > 0,$$

corresponds to dark energy in the form of quasivacuum, or cosmological constant.

The third one,

$$W_\infty > 0, \qquad \varepsilon_\infty > 0$$

is mixed: quintessence plus cosmological constant.

All three models differ in the admissible asymptotic form of the self-interaction potential of the field ϕ and the fluid mixture type. Thus if $\varepsilon_0 > 0$, then the fluid mixture must contain quasivacuum dark energy. In all three types of models, the asymptotic behavior of the Universe is the same: accelerated expansion.

Models in which $\lim_{t\to\infty} \delta = \delta_0 > 0$ correspond to different Friedmannian scenarios for which

$$W(t) \to \delta_0 t^{-2}, \qquad \varepsilon(t) \to \delta_0 t^{-2}, \qquad t \to \infty.$$

Models with $\delta < 0$ refer to phantom fields.

In all other cases $\delta \to \infty$ as $t \to \infty$, which appears to be of little interest according to to modern views. So, the present approach leads to a sufficiently simple classification of models having a physical meaning.

5. THE SELF-INTERACTION POTENTIAL

One more important aspect of the general properties of the models under consideration is the nature of the scalar field self-interaction, determined by the properties of the potential $V(\phi)$. This analysis is also important enough from the viewpoint of the restrictions that follow from the dominating energy principle. The latter implies that the potential must remain positive at all stages of the evolution. It helps one to establish additional requirements on the parameter choice for the models suggested. To this end, let us consider the self-interaction potential for models specified by the simple condition

$$\dot{W} = -Q(W - W_\infty) = -k(W - W_\infty)^\alpha, \quad 1 \leq \alpha < 3/2. \qquad (29)$$

where W_∞ is the limiting value of W as $t \to \infty$: $W_\infty = \lim_{t\to\infty} W(t)$. For the case $1 < \alpha < 3/2$ and $W_\infty = 0$ such models have been considered in [14]. In the case $\alpha = 1$, the self-interaction potential for a model with $\varepsilon \equiv 0$ was calculated in [13]. As was noted in [13, 14] and in the introduction to the present paper, the self-interaction potentials are in this case similar in shape to the Higgs potential. These circumstances play a certain role in fixing attention on the choice (29) of the function $Q(W)$ in the present paper.

Keeping in mind possible generalizations, consider the problem of calculating the potential $V(\phi)$ in the general case of two-component matter with an arbitrary function $Q(W)$. Using Eqs. (6) and (12) as well as (9) and (10), one can obtain in an explicit form an equation for the total energy $W(\phi)$ and the self-interaction potential $V(\phi)$ as functions of the field. For an arbitrary function $Q(W)$ in (29), the equation for the normalized total energy

$$w(\phi) = (W(\phi) - W_\infty)/(\varepsilon_\infty + 2W_\infty)$$

may be written in the following form:

$$\frac{dw}{d\phi} = -(3\kappa\varepsilon_0)^{1/4}(2w+1)^{1/4}(Q(w\varepsilon_0))^{1/2}. \qquad (30)$$

Here, $\varepsilon_0 = \varepsilon_\infty + 2W_\infty$. Accordingly, the expression for $v(\phi) = V(\phi)/\varepsilon_0$ will have the following general form:

$$v(\phi) = w(\phi) - \frac{1}{2\sqrt{3\kappa\varepsilon_0^3}} \frac{Q(\varepsilon_0 w)}{\sqrt{2w+1}} + w_0, \qquad (31)$$

where

$$w_0 = \frac{W_\infty}{\varepsilon_\infty + 2W_\infty} = \frac{W_\infty}{\varepsilon_0} < 1.$$

As a boundary condition for Eq. (30), one can choose the condition $w'(0) = 0$ meaning that an extremum (minimum) of the total energy coincides with the zero field value.

To choose $Q(W)$ in the form (29), $1 \leq \alpha \leq 3/2$, we additionally introduce the variable $\chi = \phi/\phi_0$, where

$$\phi_0 = (3\kappa)^{1/4}\varepsilon_0^{(2\alpha-1)/4}\sqrt{k}, \qquad 1 \leq \alpha \leq 3/2.$$

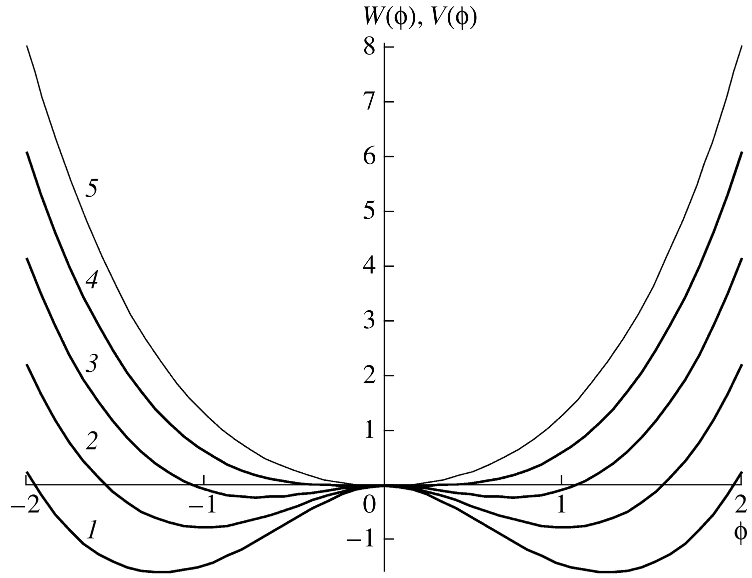

Fig. 1. The normalized total energy $w(\chi)$ (plot 5) and self-interaction potential $v(\chi)$ (plots 1-4) for $\alpha = 1$, $w_0 = 1$ and different values of ν_0: 1—$\nu_0 = 1.0$, 2—$\nu_0 = 2.0$, 3— $\nu_0 = 3.0$, 4—$\nu_0 = 4.0$.

For the case $\alpha = 1$ to be further studied in detail, it is useful to introduce the parameter Δ_0

$$\Delta_0 = k/\sqrt{3\kappa\varepsilon_0}, \tag{32}$$

which will play an important role in what follows. In this case, $\phi_0 = \sqrt{\Delta_0}(3\kappa\varepsilon_0)^{1/2}$. As a result, we obtain the following equation for $w(\chi)$:

$$\frac{dw}{d\chi} = -(2w+1)^{1/4} w^{\alpha/2}, \tag{33}$$

which does not contain parameters other than α.

For $v(\chi)$ we obtain the following equation:

$$v(\chi) = w(\chi) - \nu_0 \frac{w^\alpha(\chi)}{\sqrt{2w(\chi)+1}} + w_0, \tag{34}$$

which contains, besides α, two more dimensionless parameters w_0 and ν_0:

$$\nu_0 = \frac{k\varepsilon_0^{\alpha-3/2}}{2\sqrt{3\kappa}}.$$

For $\alpha = 1$ we have $\nu_0 = \Delta_0/2$. An analysis shows that the parameter ν_0 is responsible for the depth of minima of the function $v(\chi)$ and w_0 for the self-interaction potential values at these minima. For the case $\alpha = 1$, Fig. 1 presents plots of $w(\chi)$ and $v(\chi)$ for some values of the parameters ν_0 and $w_0 = 0$.

It is easy to see that at $\nu_0 > 0$ and $w_0 = 0$ the self-interaction potential has a range of negative values. This violates that dominating energy condition if the parameters of the Universe get into such a range. To avoid this, it is necessary to choose the value of w_0 in a special way. For $\alpha = 1$, the minima of the potential are determined by a real solution of the equation

$$(2w_m + 1)^3 = \nu_0^2 (1 + w_m)^2,$$

where w_m is the full normalized field energy value at the minimum. It follows that to fulfil the condition $V(\phi) \geq 0$ for all ϕ the value of w_0 should satisfy the inequality

$$w_0 > w_m \left(\frac{\nu_0^{2/3}}{(1+w_m)^{1/3}} - 1 \right).$$

6. DYNAMICS OF MODELS WITH A MASTER FIELD AND $1 \leq \alpha \leq 3/2$

A solution of Eq. (29) may be written in the following general form:

$$W(t) = \begin{cases} W_0 t^{-m} + W_\infty, & 1 < \alpha \leq 3/2; \\ W_0 e^{-kt} + W_\infty, & \alpha = 1. \end{cases} \tag{35}$$

Here, $m = (\alpha - 1)^{-1}$ and W_0 is an integration constant. Other values of α lead to exotic models.

Using (35) and (26), we obtain:

$$\varepsilon(t) = \begin{cases} W_0 t^{-m} + \varepsilon_\infty + W_\infty, & 1 < \alpha \leq 3/2; \\ W_0 e^{-kt} + \varepsilon_\infty + W_\infty, & \alpha = 1. \end{cases}$$

In what follows, it is convenient to introduce, instead of W_0, the dimensionless parameter

$$\xi_0 = W_0/\varepsilon_0.$$

Using (27), we find an expression for δ:

$$\delta(t) = \begin{cases} \dfrac{m\varepsilon_0^{-1/2}}{\sqrt{3\kappa}} \dfrac{\xi_0 t^{-m-1}}{(1 - w_0 + \xi_0 t^{-m})\sqrt{2\xi_0 t^{-m} + 1}}, \\ 1 < \alpha \le 3/2; \\ \dfrac{k\varepsilon_0^{-1/2}}{\sqrt{3\kappa}} \dfrac{\xi_0 e^{-kt}}{(1 - w_0 + \xi_0 e^{-kt})\sqrt{2\xi_0 e^{-kt} + 1}}, \\ \alpha = 1. \end{cases} \quad (36)$$

As a function of H, the parameter δ for these models has the following form:

$$\delta(t) = \begin{cases} \dfrac{2k\kappa^{1-\alpha}}{3H} \dfrac{(3H^2 - \kappa W_\infty)^\alpha}{(3H^2 - \kappa\varepsilon_\infty)}, \\ 1 < \alpha \le 3/2; \\ \dfrac{2k}{3H} \dfrac{(3H^2 - \kappa W_\infty)}{(3H^2 - \kappa\varepsilon_\infty)}, \\ \alpha = 1. \end{cases} \quad (37)$$

Further we accordingly find

$$R(t) = \begin{cases} R_0 \exp\left\{\sqrt{\kappa/3}\varepsilon_0^{1/2} \displaystyle\int \sqrt{2\xi_0 t^{-m} + 1} dt\right\}, \\ 1 < \alpha \le 3/2; \\ R_0 \exp\left\{\sqrt{\kappa/3}\varepsilon_0^{1/2} \displaystyle\int \sqrt{2\xi_0 e^{-kt} + 1} dt\right\}, \\ \alpha = 1. \end{cases} \quad (38)$$

and

$$T(t) = \begin{cases} \dfrac{R^3(t) m\varepsilon_0^{1/2}}{s_0\sqrt{3æ}} \dfrac{\xi_0 t^{-m-1}}{\sqrt{2\xi_0 t^{-m} + 1}}, \\ 1 < \alpha \le 3/2; \\ \dfrac{R^3(t) k\varepsilon_0^{1/2}}{s_0\sqrt{3æ}} \dfrac{\xi_0 e^{-kt}}{\sqrt{3æ}\sqrt{2\xi_0 e^{-kt} + 1}}, \\ \alpha = 1. \end{cases} \quad (39)$$

7. ANALYSIS OF MODELS

7.1. End of Inflation

As was already pointed out, models with $1 < \alpha < 3/2$ have been considered in [14]. Although here we have obtained exact analytic solutions for these modes all their basic properties were revealed in the cited paper. Let us therefore concentrate on the model with $\alpha = 1$, comparing it, where necessary with models $1 < \alpha < 3/2$, to be called power-law models in what follows.

Let us above all notice some features of the model distinguishing it from models with $1 < \alpha < 3/2$. First, as is easily seen from solutions for the scale factor and other parameters, a cosmological singularity at $\alpha = 1$ is located at minus infinity in time, $t \to -\infty$. Unlike that, a cosmological singularity at $1 < \alpha < 3/2$ is located at the point $t = 0$, where where all energy parameters (except δ) turn to infinity. Meanwhile, the model $\alpha = 1$ has, like the power-law models, a time interval which can be naturally called inflation. The inflationary epoch (see Fig.) in this model actually begins at $t \to -\infty$ and ends somewhat near the instant at which δ has an extremum (Fig. 2). More precisely the moment when inflation terminates, as shown in [13, 14], may be related to the moment when the velocity of sound in the matter components becomes real, so that perturbations can freely propagate in the Universe. Since the velocity of sound squared, expressed in terms of the velocity of light, is equal for a fluid to $\gamma = \delta - 1$, termination of inflation may be attributed to the moment of time t_i when $c_\varepsilon^2 = \gamma(t_i) = 0$. This moment may be found by solving Eqs. (36) for the value $\delta = 1$. For the field component, the velocity of sound may be found from Eq. (9). The corresponding relation has the form

$$C_W^2 = -1 + \frac{1}{\sqrt{3\kappa}} \frac{Q(W)}{\sqrt{2W + \varepsilon_\infty}} \times \left[\frac{d\ln Q}{dW} - \frac{1}{2W + \varepsilon_\infty}\right].$$

This expression is somewhat different from (36), and consequently inflation ends, in general, at different moments for the two components of matter. Let us also note that a value of the effective parameter δ_W: $\mathcal{P} = \delta_W W$, has the following form:

$$\delta_W = -1 + \frac{1}{\sqrt{3\kappa}} \frac{Q(W)}{W\sqrt{2W + \varepsilon_\infty}}.$$

It is easy to estimate that for $\alpha = 1$, as $t \to -\infty$, and for $1 < \alpha < 3/2$ as $t \to 0$, the quantity C_W^2 behaves precisely as $c_\varepsilon^2 = \gamma(t)$, i.e., tends to -1. As $t \to \infty$, we have

$$C_W^2(\infty) = -1 + \frac{1}{\sqrt{3\kappa\varepsilon_0}} \left[\alpha - \frac{W_\infty}{\varepsilon_0}\right].$$

It is seen from this relation that the scalar field behavior at large times may differ from that of the fluid for which always $\gamma(\infty) = -1$.

7.2. Radiation-Dominated Epoch

The instant t_* when an extremum of δ is achieved is calculated from the relation (36) and has the form

$$t_* = -\frac{1}{k} \ln\left[\frac{\varepsilon_0}{W_0}\right] - \ln c_0, \quad (40)$$

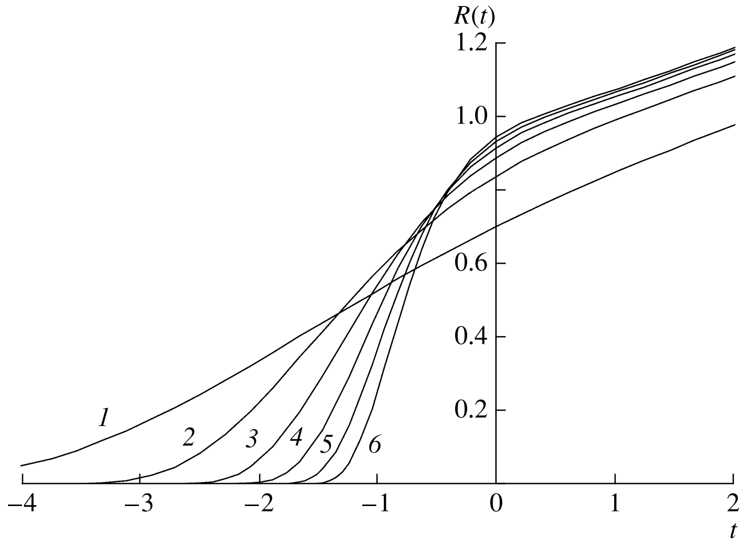

Fig. 2. Evolution of the scale factor in the model $\alpha = 1$, $W_0 = 1$, $\varepsilon_0 = 0.5$ for different k: 1—$k = 1$, 2—$k = 2$, 3— $k = 3$, 4—$k = 4$, 5—$k = 5$, 6—$k = 6$.

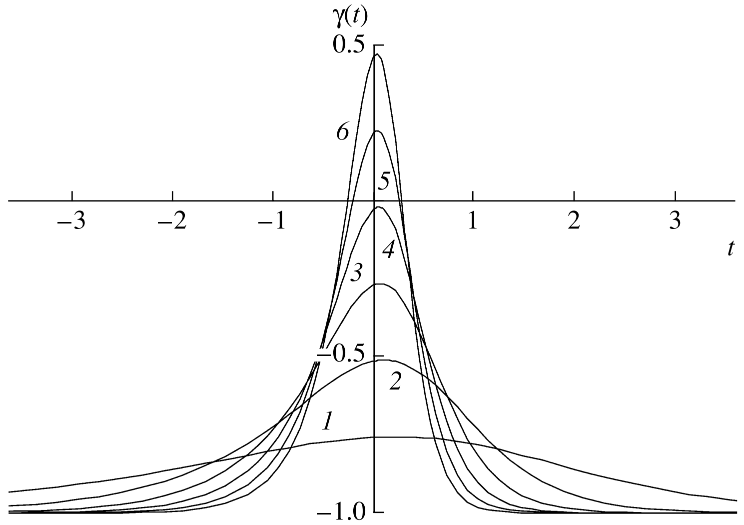

Fig. 3. Evolution of the parameter γ in the model $\alpha = 1$, $W_0 = 1$, $\varepsilon_0 = 0.5$ for different k: 1—$k = 1$, 2— $k = 2$, 3—$k = 3$, 4—$k = 4$, 5—$k = 5$, 6—$k = 6$.

where $c_0 = \left[1 + \sqrt{5}\right]/2 \simeq 1.618$.

The exremum value $\delta_* = \delta(t_*)$ itself is equal to

$$\delta_* = \delta_0 \frac{k}{\sqrt{3\kappa\varepsilon_0}} = \delta_0 \Delta_0, \tag{41}$$

where δ_0 is a numerical parameter:

$$\delta_0 = \frac{c_0}{(c_0 + 1)\sqrt{2c_0 + 1}} \simeq 0.3.$$

Analyzing the behavior of the curve $\delta(t)$, one can conclude that the inflation time is basically determined by the parameter k, i.e., the characteristic time τ of ϕ field decrease by a factor of e: $\tau = 1/k$.

According to the general modern views on the Universe evolution, there was an epoch dominated by radiation in the form of electromagnetic waves in thermal equilibrium with the rest of matter. Domination of isotropic radiation means that the quantity $\gamma_r \simeq 1/3$ while δ_* should be maximally close to $\delta_r = 1 + \gamma_r = 4/3$. From these considerations one can find some estimates of the model parameters. From (41),

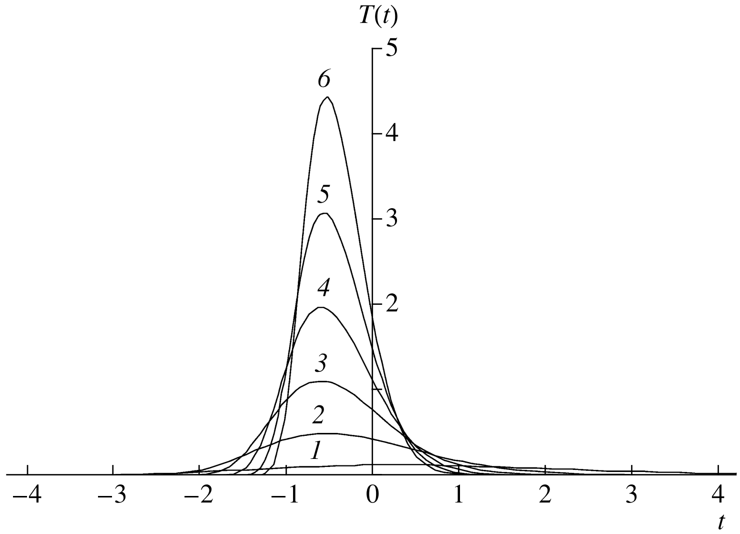

Fig. 4. Temperature evolution in the model $\alpha = 1$, $W_0 = 1$, $\varepsilon_0 = 0.5$ and different k (under the condition $\delta_* > q_0$): 1—$k = 1$, 2—$k = 2$, 3—$k = 3$, 4—$k = 4$, 5—$k = 5$, 6—$k = 6$.

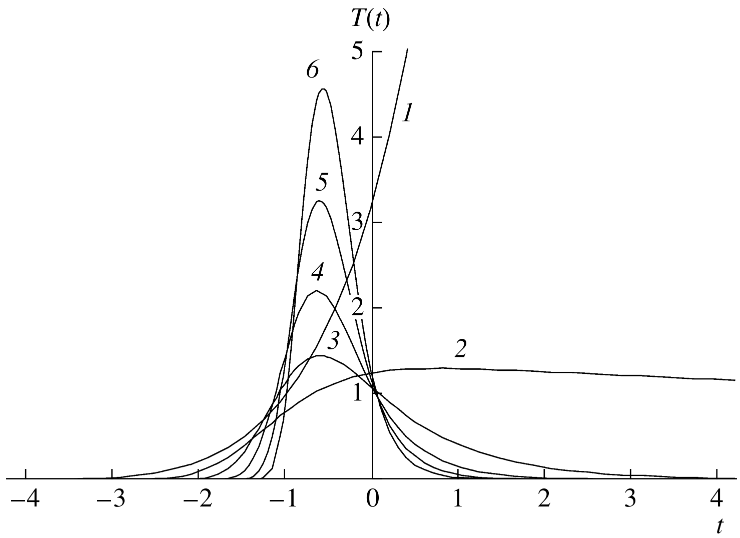

Fig. 5. Temperature evolution in the model $\alpha = 1$, $W_0 = 1$, $\varepsilon_0 = 20.5$ and different k: 1—$k = 1$, 2—$k = 2$, 3—$k = 3$, 4—$k = 4$, 5—$k = 5$, 6—$k = 6$.

assuming $\delta_* = 4/3$, we find

$$\Delta_0 = \frac{k}{\sqrt{3\kappa\varepsilon_0}} = 4/(3\delta_0) = \frac{(c_0+1)\sqrt{2c_0+1}}{3c_0} \simeq 4.4 > 1. \tag{42}$$

Hence we obtain an estimate for k:

$$k\Big|_{\delta_*=1/3} = 4.4\frac{(c_0+1)\sqrt{2c_0+1}}{3c_0}\sqrt{3\kappa\varepsilon_0} = 4.4 \cdot \sqrt{3\kappa\varepsilon_0}. \tag{43}$$

7.3. The Modern Epoch

One of the most important new elements that appear in the model with $\alpha = 1$ as compared with the power-law models is a different behavior of the temperature as $t \to \infty$. As was shown in [14] on the basis of an asymptotic analysis, and in the present paper follows directly from the exact relations (38) and (39), the temperature in models with $1 < \alpha < 3/2$ asymptotically grows for $\varepsilon_0 > 0$ exponentially due to an exponential growth of the scale factor. The power-law decrease of the energy parameters cannot prevent this growth. Unlike that, in the case $\alpha = 1$, the models split into three classes. As is easily seen from the relations (38) and (39), at $\alpha = 1$ the asymptotic behavior of the temperature as $t \to \infty$ is determined by the exponential factor with the exponent $\lambda = \sqrt{3\kappa\varepsilon_0} - k$. Hence it follows that for $\lambda > 0$, which is equivalent to $\sqrt{3\kappa\varepsilon_0} > k$ or $\Delta_0 < 1$, the temperature exponentially grows along with the scale factor. In the case $\lambda < 0$ or $\Delta_0 > 1$, the temperature exponentially decreases as $t \to \infty$, and for $\lambda = 0$ or $\Delta_0 = 1$ it tends to a constant value T_∞:

$$T_\infty = \frac{R_0^3 k \xi_0 \varepsilon_0^{1/2}}{s_0 \sqrt{3æ}}.$$

Following the estimate (42), we see that the temperature decrease as $t \to \infty$ just corresponds to the condition $\delta_* > 1/3$. In other words, the temperature in these models will decrease with time if the Universe has undergone an epoch when $\gamma_* = \delta_* - 1 > -2/3$. Otherwise the temperature in it would have grown exponentially. Passing of the Universe through a radiation-dominated epoch is an indicator of a subsequent temperature decrease in the modern epoch.

Different types of temperature behavior are presented in Figs. 4 and 5. Fig. 4 presents the time dependence of the temperature for the case $\Delta_0 > 1$ at different values of k corresponding to this requirement. Fig. 5 illustrates the temperature behavior for the value $k = 1$, such that the condition $\Delta_0 < 1$ holds, while for $k = 2$ we have $\Delta_0 \simeq 1$. For other values of k, the parameter $\Delta_0 > 1$.

8. CONCLUSION

The above analysis makes it possible to come to the following conclusions. First, the models presented a master field presented here, in the zero order, describe the behavior of the basic parameters of the Universe agreeing with the modern data, at least for the considered types of evolution of the master scalar field total energy. For a more detailed analysis, it is necessary to consider the evolution of density perturbations in such models for two-component matter, field + fluid. Second, the method of model analysis described here and the explicit relations obtained allow for analyzing models with any decrease rate of the scalar field energy, including also different types with an exponentially decreasing energy. Third, the models presented can easily be classified according to the matter type that provides the modern accelerated expansion. This can be done from the type of asymptotic behavior of the total energy of the whole matter, i.e., field and fluid. Fourth, an analysis of these models shows that to provide the observed decrease in the mean temperature of matter in the Universe it is necessary to have an exponential regime of the field energy decrease, at least, in the modern epoch. The latter means that the self-interaction potential must have a shape close to that of the Higgs potential.

As a whole, one can assert that the models presented make it possible to take into account the data on the Universe expansion in a sufficiently flexible manner and thus improve the model parameters. A distinctive feature of the models studied here and in [14] is that, unlike the Standard model, the temperature here grows during the inflationary period from zero to a certain maximum value, and the moment when it is reached actually coincides with the end of inflation. Such a behavior appears to be quite logical if one takes into account that there is no normal matter during inflation. This distinction requires a further study from the viewpoint of evolution of perturbation spectra in this period, which may answer the question of a realistic nature of these scenarios.

Appendix

THERMODYNAMIC PARAMETERS OF THE SCALAR FIELD

To justify the possible interpretation of the effective thermodynamic parameters of the scalar field introduced in this paper as real thermodynamic parameters of the system, let us show that these parameters can be related to statistical parameters of scalar field fluctuations in the framework of the model under consideration.

Let us present the scalar field ϕ as an expansion, $\phi = \Phi + \phi'$, where $\Phi = \Phi(t)$ is the ensemble mean value of the scalar field, $\Phi = \langle\phi\rangle$, and ϕ' is the field fluctuation with a zero expectation value, $\langle\phi'\rangle = 0$. Here and henceforth the angular brackets $\langle\rangle$ denote ensemble averaging. The mean value $\overline{E}$ of the field energy density and that of its effective pressure, $\overline{P}$, can in this case be represented as follows:

$$\overline{E} = \langle E(\Phi + \phi')\rangle = \frac{1}{2}\dot\Phi^2 + E(\Phi) + \frac{1}{2}\langle\dot{\phi'}^2\rangle + \sum_{k=1}^{\infty}\frac{1}{k!}\frac{d^k E(\Phi)}{d\Phi^k}\langle\phi'^k\rangle, \qquad (A.1)$$

$$\overline{P} = \langle P(\Phi + \phi')\rangle = \frac{1}{2}\dot\Phi^2 - E(\Phi) + \frac{1}{2}\langle\dot{\phi'}^2\rangle - \sum_{k=1}^{\infty}\frac{1}{k!}\frac{d^k E(\Phi)}{d\Phi^k}\langle\phi'^k\rangle. \qquad (A.2)$$

The first of these relations may be considered as an integrated analogue of the first law of thermodynamics for the scalar field:

$$\Delta\mathcal{E} = \Delta\mathcal{W} + \Delta\mathcal{A} \qquad (A.3)$$

The increment of the total field energy in the volume V splits into two components: $\Delta\mathcal{A}$, the work that changes the energy of the mean field and the increment of the field internal energy (the fluctuation energy):

$$\mathcal{U} = \frac{1}{2}\langle\dot{\phi'}^2\rangle + \sum_{k=1}^{\infty}\frac{1}{k!}\frac{d^kE(\Phi)}{d\Phi^k}\langle\phi'^k\rangle.$$

Therefore the system may be ascribed an entropy $\mathcal{S}$ and a temperature Θ. This may be done by directly representing the total field energy increment in the volume V in the form

$$\Delta\mathcal{E} = \Theta\, d\mathcal{S}.$$

Or, otherwise, one can do that directly from the equation of state $\overline{P} = F(\overline{E})$ which directly follows from (A.1) and (A.2) since all functions involved in these equations are only functions of time. It is well known that if the processes are required to be reversible, we have two relations for the entropy:

$$\Theta\frac{\partial\mathcal{S}}{\partial\Theta} = \frac{\partial\mathcal{W}}{\partial\Theta},$$

$$\Theta\frac{\partial\mathcal{S}}{\partial V} = \frac{\partial\mathcal{W}}{\partial V} + \overline{P} = \overline{E} + \overline{P}. \qquad \text{(A.4)}$$

These two relations, provided there is an equation of state, make it possible to introduce both the entropy $\mathcal{S}$ and the temperature Θ. Hence we obtain the relation

$$\overline{E} = \frac{dU}{dV} = -\overline{P} + \Theta\frac{d\overline{P}}{d\Theta}.$$

It is this relation that was used in the paper, with the difference that we kept in mind the ensemble averaged values of the system parameters under the identifications $\mathcal{P} = \overline{P}$ and $W = \overline{E}$..

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